CONTENTS

Chapter – 1

DISCRETE TIME SIGNALS AND SYSTEMS

1.1 Signal and Types

A *signal* is defined as any physical quantity that varies with time, space or any other independent variables. Mathematically, a signal is described as a function of one or more independent variables which gives the behaviour of a phenomenon. Examples: Speech signal, image signal, $x = \sin\omega t$, etc.

Types of Signal

i. **Continuous time signal:** A signal which can be defined in every instance of time under consideration is known as *continuous time signal*. It is represented as x(t), where t is continuous and can assume values in the range $-\infty \leq t \leq \infty$.

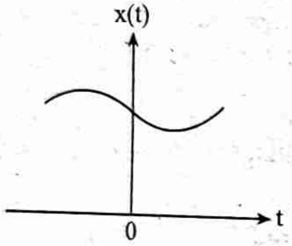


Fig.: Continuous time signal x(t)

ii. **Discrete time signal:** A signal which is defined only at certain time instants is known as *discrete time signal*. It is represented as x[n], where n assumes discrete values. If we quantize x[n], we obtain digital signal.

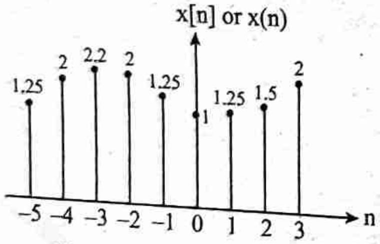


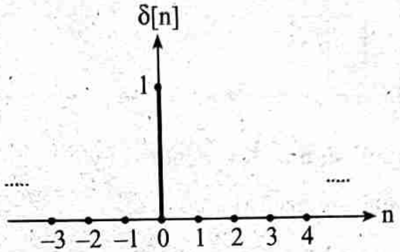
Fig.: Discrete time signal x[n]

Basic Signals Types

i. **Unit impulse signal (dirac delta function):** It is denoted by δ[n] and is defined as

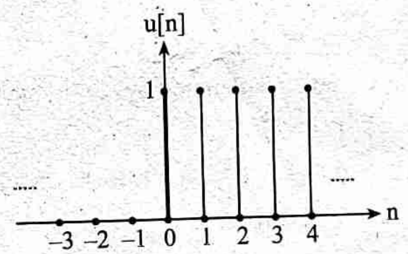
$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Unit impulse signal is a signal that is zero everywhere, except at n = 0 where its value is unity.



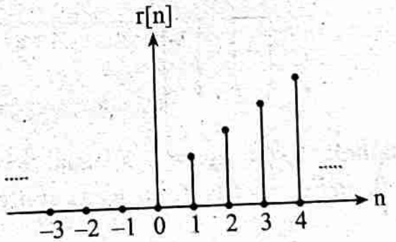
ii. **Unit step signal:** It is denoted by u[n] and is defined as

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



iii. **Unit ramp signal:** It is denoted by r[n] and is defined as

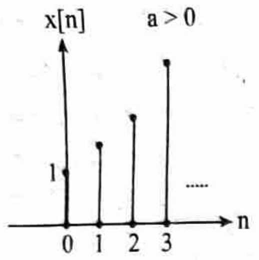
$$r[n] = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

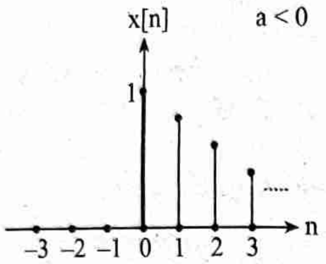


iv. **Exponential signal:** It is denoted by x[n] and is defined as $x[n] = ce^{an}$; where c & a are real numbers.

Here, if a > 0, it is growing exponential

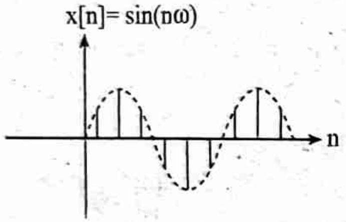
if a < 0, it is decaying exponential





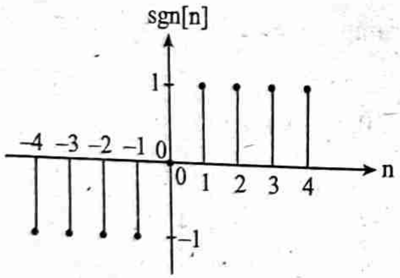
v. **Sinusoidal signal:** It is denoted by x[n] and is defined as

x[n] = sin(nω) or x[n] = cos[nω]



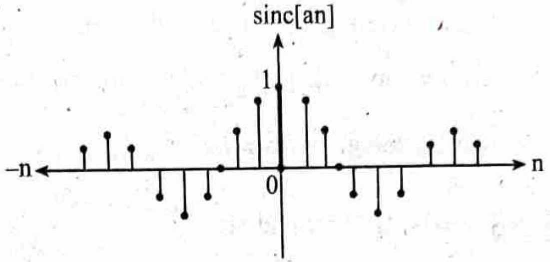
vi. **Signum signal:** It is denoted by sgn[n] and is defined as

$$\text{sgn}[n] = \begin{cases} 1. \ n < 0 \\ 0, \ n = 0 \\ -1, \ n < 0 \end{cases}$$



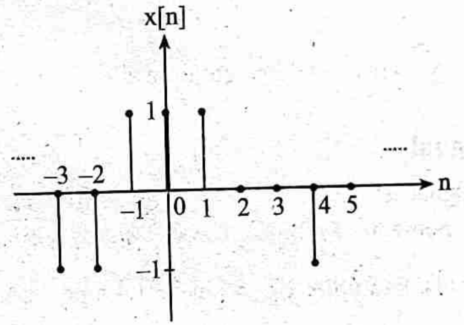
vii. **Sinc signal:** A *sinc signal* is a signal that gives unity value at n = 0, and decays as it moves away from the origin. It is defined as:

$$\text{sinc}[an] = \begin{cases} 1 & ,n = 0 \\ \dfrac{\sin(an)}{an}, & \text{elsewhere} \end{cases}$$



Representation of Discrete Time Signalg

i. **Graphical representation:**



ii. **Functional representation:**

$$x[n] = \begin{cases} 1, -1 \le n \le 1 \\ -1, \ n = -3, -2, 4 \\ 0, \text{ elsewhere} \end{cases}$$

iii. **Tabular representation:**

n	....	–3	–2	–1	0	1	2	3	4	5	....
x[n]	....	–1	–1	1	1	1	0	0	–1	0	....

iv. **Sequence representation:**

$$x[n] = \{\ldots\ldots -1, -1, 1, \underset{\uparrow}{1}, 1, 0, 0, -1, 0, \ldots\ldots\}$$

1.2 Energy Signal and Power Signal

Energy Signal

A signal is said to be *energy signal* if it has finite non-zero energy and zero average power. Usually aperiodic signals are energy signals. Example: $x[n] = \left(\frac{1}{4}\right)^n u[n]$ is an energy signal.

To calculate energy of a signal:

$$E = \int_{-\infty}^{\infty} x^2(t)\, d(t) \quad ; \text{for real } x(t)$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt \quad ; \text{for complex } x(t)$$

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 \quad ; \text{discrete time signal}$$

Power Signal

A signal is said to be *power signal* if it has infinite energy and finite non-zero average power. Usually periodic signals are power signals. Example: $x[n] = \sin\left(\frac{\pi n}{3}\right)$ is a power signal.

To calculate average power of a signal:

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt \quad ; \text{for real } x(t)$$

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt \quad ; \text{for complex } x(t)$$

$$P = \lim_{N \to \infty} \frac{1}{N+1} \sum_{n=-N/2}^{N/2} |x[n]|^2 \quad ; \text{discrete time signal}$$

$$\text{or, } P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

Example 1.1:

Determine whether the given signal is energy or power signal.

$$\mathbf{x[n] = \{3, 1, 0, \underset{\uparrow}{2+2j}, 7\}}$$

Solution:

Given, signal is

$$x[n] = \{3, 1, 0, \underset{\uparrow}{2+2j}, 7\}$$

Energy of the given signal,

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= \sum_{n=-2}^{2} |x[n]|^2$$

$$= (3)^2 + (1)^2 + (0)^2 + (|2+2j|)^2 + (7)^2$$

$$= 9 + 1 + 0 + (\sqrt{2^2+2^2}\,)^2 + 49$$

$$= 59 + 4 + 4 = 67 \text{ Joule}$$

Average power of the given signal,

$$P_{avg} = \lim_{N \to \infty} \frac{1}{N+1} \sum_{n=-N/2}^{N/2} |x[n]|^2$$

$$= \lim_{N \to \infty} \frac{1}{4+1} \sum_{n=-2}^{2} |x[n]|^2$$

$$= \lim_{N \to \infty} \frac{1}{5} \times 67 = 13.4 \text{ Watts}$$

Since both the energy and average power of the given signal is of finite value, it is neither energy signal nor power signal.

Example 1.2:

Determine whether the given signal is energy or power signal.

$$\mathbf{x[n] = \left(\frac{1}{4}\right)^n u[n]}$$

Solution:

Given signal is, $x[n] = \left(\frac{1}{4}\right)^n u[n]$

Energy of the given signal,

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= \sum_{n=-\infty}^{\infty} \left[\left(\frac{1}{4}\right)^n u[n]\right]^2$$

$$= \sum_{n=-\infty}^{-1} \left[\left(\frac{1}{4}\right)^n u[n]\right]^2 + \sum_{n=0}^{\infty} \left[\left(\frac{1}{4}\right)^n u[n]\right]^2$$

We know,

$$u[n] = \begin{cases} 1, n \geq 0 \\ 0, n < 0 \end{cases}$$

$$\text{So, } E = 0 + \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{2n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n$$

$$= \frac{1}{1 - 1/16} \qquad \left[\because \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \text{ for } |a| < 1\right]$$

$$\therefore\ E = \frac{16}{15}$$

The energy of the given signal is finite and non zero.

Average power of the given signal,

$$P_{avg} = \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=-N/2}^{N/2} |x[n]|^2$$

$$= \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} \left[\left(\frac{1}{4}\right)^n u[n]\right]^2$$

$$= \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} \left(\frac{1}{4}\right)^{2n}$$

$$= \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} \left(\frac{1}{16}\right)^n$$

$$= \lim_{N\to\infty} \frac{1}{N+1} \left[\frac{1-(1/16)^{N+1}}{1-(1/16)}\right] \left[\because \sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}\right]$$

$$= \lim_{N\to\infty} \frac{1}{N+1} \times \frac{1}{15/16} \times \left[1 - \frac{1}{16^{N+1}}\right]$$

$$= \lim_{N\to\infty} \frac{1}{N+1} \times \frac{16}{15} - \lim_{N\to\infty} \frac{1}{N+1} \times \frac{16}{15} \times \frac{1}{16^{N+1}}$$

$$= 0 - 0$$

$$\therefore\ P_{avg} = 0$$

The average power of the given signal is zero.

Hence, the given signal $x[n] = \left(\frac{1}{4}\right)^n u[n]$ is energy signal.

Example 1.3:

Determine whether the given signal is energy or power signal.

$$\mathbf{x[n] = \sin\left(\frac{\pi}{3n}\right)}$$

Solution:

Given signal is, $x[n] = \sin\left(\frac{\pi}{3n}\right)$

Energy of the given signal,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} \left|\sin\left(\frac{\pi}{3n}\right)\right|^2$$

$$= \sum_{n=-\infty}^{\infty} \sin^2\left(\frac{\pi}{3n}\right)$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1-\cos\left(\frac{2\pi}{3n}\right)}{2}\right]$$

$$= \frac{1}{2} \sum_{n=-\infty}^{\infty} \left[1 - \cos\left(\frac{2\pi}{3n}\right)\right]$$

$$=\frac{1}{2}\sum_{n=-\infty}^{\infty}1-\frac{1}{2}\sum_{n=-\infty}^{\infty}\cos\left(\frac{2\pi}{3n}\right)$$

We have,

$$\sum_{n=-\infty}^{\infty}1=\infty$$

and, $\sum_{n=-\infty}^{\infty}\cos\left(\frac{2\pi n}{3}\right)\approx 0$

∵ For n = 0; $\cos\left(\frac{2\pi n}{3}\right)=1$

For n = 1; $\frac{-1}{2}$

For n = 2; $\frac{-1}{2}$

For n = 3; 1

For n = 4; $\frac{-1}{2}$

For n = 5; $\frac{-1}{2}$

and so on.

So, $1-\frac{1}{2}-\frac{1}{2}+1-\frac{1}{2}-\frac{1}{2}\ldots\ldots\ldots\approx 0$

∴ $E=\infty$

Average power of the given signal,

$$P_{avg}=\lim_{N\to\infty}\frac{1}{N+1}\sum_{n=-N/2}^{N/2}|x[n]|^2$$

$$=\lim_{N\to\infty}\frac{1}{N+1}\sum_{n=-N/2}^{N/2}\left[\sin\left(\frac{\pi n}{3}\right)\right]^2$$

$$=\lim_{N\to\infty}\frac{1}{N+1}\sum_{n=-N/2}^{N/2}\sin^2\left(\frac{\pi n}{3}\right)$$

$$=\lim_{N\to\infty}\frac{1}{N+1}\sum_{n=-N/2}^{N/2}\left[\frac{1-\cos\left(\frac{2\pi n}{3}\right)}{2}\right]$$

$$=\lim_{N\to\infty}\frac{1}{N+1}\times\frac{1}{2}\left[\sum_{n=-N/2}^{N/2}1-\sum_{n=-N/2}^{N/2}\cos\left(\frac{2\pi n}{3}\right)\right]$$

We have,

$$\sum_{n=-N/2}^{N/2}1=N+1$$

and, $\sum_{n=-N/2}^{N/2}\cos\left(\frac{2\pi n}{3}\right)\approx 0$

$$\therefore\ P_{avg}=\lim_{N\to\infty}\frac{1}{N+1}\times\frac{1}{2}\times(N+1-0)$$

$$=\lim_{N\to\infty}\frac{1}{N+1}\times\frac{1}{2}\times(N+1)$$

$$\therefore\ P_{avg}=\frac{1}{2}$$

The given signal $x[n]=\sin\left(\frac{\pi n}{3}\right)$ is a power signal since it has infinite energy and finite non-zero average power.

1.3 Periodicity of Discrete Time Signal

Periodic Signal

A signal is said to be *periodic* if it repeats itself at a fixed interval of time. For periodic signal,

$x[n]=x[n+N]$; where N is period and always integer.

Aperiodic Signal

A signal is said to be *aperiodic* if it does not repeat itself after a fixed interval of time. For aperiodic signal,

$x[n]\neq x[n+N]$

Aperiodic signals are assumed to repeat themselves at infinity.

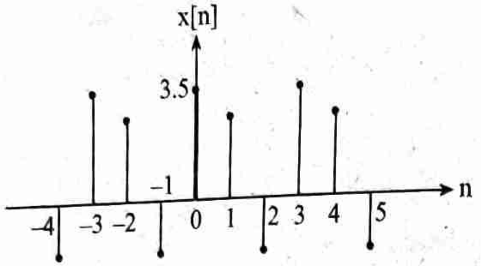


Fig.: A digital periodic signal x[n] with period N = 3

Condition for a DT Signal to be Periodic

Let us consider a discrete time (DT) signal x[n] having period N. For this signal to be periodic, x[n] = x[n + N] needs to be satisfied.

Suppose, $x[n] = A\cos(2\pi f_0 n + \theta)$,

Then,

$$x[n+N] = A\cos(2\pi f_0(n+N) + \theta)$$

$$= A\cos(2\pi f_0 n + 2\pi f_0 N + \theta)$$

For periodicity,

$$x[n] = x[n+N]$$

or, $A\cos(2\pi f_0 n + \theta) = A\cos(2\pi f_0 n + 2\pi f_0 N + \theta)$

To satisfy above condition, $2\pi f_0 N = 2\pi k$ must be satisfied; where k is an integer [$\because \cos(2\pi k + A) = \cos(A)$].

On solving, we get,

$$f_0 N = k$$

$$\Rightarrow f_0 = \frac{k}{N}$$

Hence, a discrete time signal is periodic if its fundamental frequency (f_0) is a ratio of two integers.

For $x[n] = x_1[n] + x_2[n]$ to be periodic, both $x_1[n]$ and $x_2[n]$ must be periodic. Also, the period of x[n] is LCM of period of $x_1[n]$ and $x_2[n]$.

Example 1.4:

Check the periodicity for the given signal and find its period.

x[n] = cos3n

Solution:

Given signal is, x[n] = cos3n .... (i)

We know, for a discrete time signal, the signal is periodic if its fundamental frequency is a ratio of two integers.

Comparing (i) with $x[n] = \cos\omega_0 n$, we have

$$\omega_0 = 3$$

or, $2\pi f_0 = 3$

$$\Rightarrow f_0 = \frac{3}{2\pi}$$

Since the fundamental frequency is not a ratio of two integers, the given signal is not periodic.

Example 1.5:

Check the periodicity for the given signal and find its period.

x[n] = cos21πn

Solution:

Given signal is, x[n] = cos21πn .... (i)

We know, for a discrete time signal, the signal is periodic if its fundamental frequency is a ratio of two integers.

Comparing (i) with $x[n] = \cos\omega_0 n$, we have

$$\omega_0 = 21\pi$$

or, $2\pi f_0 = 21\pi$

or, $f_0 = \frac{21\pi}{2\pi} = \frac{21}{2} = \frac{k}{N}$

Since the fundamental frequency is a ratio of two integers, the given signal is periodic.

The period of the given signal is N = 2.

Example 1.6:

Check the periodicity for the given signal and find its period.

$x[n] = \cos 21\pi n + \cos n$

Solution:

Given signal is, $x[n] = \cos 21\pi n + \cos n$ .... (i)

We know, for a DT signal, the signal is periodic if its fundamental frequency is a ratio of two integers.

Also, for $x[n] = x_1[n] + x_2[n]$ to be periodic, both $x_1[n]$ and $x_2[n]$ must be periodic.

Comparing (i) with $x[n] = x_1[n] + x_2[n]$, we have

$x_1[n] = \cos 21\pi n$

$x_2[n] = \cos n$

Case-I: $x_1[n] = \cos 21\pi n$

Comparing with $x[n] = \cos\omega_0 n$, we have

$\omega_0 = 21\pi$

or, $2\pi f_0 = 21\pi$

or, $f_0 = \frac{21\pi}{2\pi} = \frac{21}{2} = \frac{k}{N_1}$

Since the fundamental frequency is a ratio of two integers, $x_1[n]$ is periodic.

Case-II: $x_2[n] = \cos n$

Comparing with $x[n] = \cos\omega_0 n$, we have

$\omega_0 = 1$

or, $2\pi f_0 = 1$

$\Rightarrow f_0 = \frac{1}{2\pi}$

Since the fundamental frequency is not a ratio of two integers, $x_2[n]$ is not periodic.

Hence, $x[n] = \cos 21\pi n + \cos n$ is not periodic since $\cos n$ is not periodic.

Example 1.7:

Determine if the signal $x[n] = \cos\left(\frac{2\pi n}{5}\right) + \sin\left(\frac{\pi n}{3}\right)$ is periodic or not. If the signal is periodic, find its fundamental period. **[2081 Bhadra]**

Solution:

$x[n] = \cos\left(\frac{2\pi n}{5}\right) + \sin\left(\frac{\pi n}{3}\right)$

Comparing this with $x[n] = x_1[n] + x_2[n]$, we have

$x_1[n] = \cos\left(\frac{2\pi n}{5}\right)$

$x_2[n] = \sin\left(\frac{\pi n}{3}\right)$

Case-I: $x_1[n] = \cos\left(\frac{2\pi n}{5}\right)$

Comparing with $x[n] = \cos\omega_0 n$, we have

$\omega_0 = \frac{2\pi}{5}$

or, $2\pi f_0 = \frac{2\pi}{5}$

$\Rightarrow f_0 = \frac{2\pi}{5} \times \frac{1}{2\pi} = \frac{1}{5} = \frac{k}{N}$

Since the fundamental frequency is a ratio of two integers, $x_1[n]$ is periodic.

And, period of $x_1[n] = 5$.

Case-II: $x_2[n] = \sin\left(\frac{\pi n}{3}\right)$

Comparing with $x[n] = \sin\omega_0 n$, we have

$\omega_0 = \frac{\pi}{3}$

or, $2\pi f_0 = \frac{\pi}{3}$

$$\Rightarrow f_0 = \frac{\pi}{3} \times \frac{1}{2\pi} = \frac{1}{6} = \frac{k}{N}$$

Since the fundamental frequency is a ratio of two integers, $x_2[n]$ is periodic.

And, period of $x_2[n] = 6$

Hence, the given signal $x[n] = \cos\left(\frac{2\pi n}{5}\right) + \sin\left(\frac{\pi n}{3}\right)$ is periodic since both $\cos\left(\frac{2\pi n}{5}\right)$ and $\sin\left(\frac{\pi n}{3}\right)$ are periodic.

Fundamental period of x[n]

= LCM of period of $x_1[n]$ and $x_2[n]$

= 30

Example 1.8:

Determine whether the given signal is periodic or not. If the signal is periodic, determine the fundamental period:

$x[n] = e^{\frac{j\pi n}{16}} \cos\left\{\frac{n\pi}{17}\right\}$. **[2080 Bhadra]**

Solution:

Given signal is, $x[n] = e^{\frac{j\pi n}{16}} \cos\left\{\frac{n\pi}{17}\right\}$

$$\text{or, } x[n] = \left[\cos\left(\frac{\pi n}{16}\right) + j\sin\left(\frac{\pi n}{16}\right)\right]\cos\left(\frac{n\pi}{17}\right)$$

$$= \cos\left(\frac{\pi n}{16}\right) + \cos\left(\frac{n\pi}{17}\right) + j\sin\left(\frac{\pi n}{16}\right)\cos\left(\frac{n\pi}{17}\right)$$

$$= \frac{1}{2} \times 2\cos\left(\frac{\pi n}{16}\right)\cos\left(\frac{n\pi}{17}\right) + \frac{1}{2} \times 2\sin\left(\frac{\pi n}{16}\right)\cos\left(\frac{n\pi}{17}\right)$$

$$= \frac{1}{2}\left[\cos\left(\frac{\pi n}{16} + \frac{n\pi}{17}\right) + \cos\left(\frac{\pi n}{16} - \frac{n\pi}{17}\right)\right] +$$

$$\frac{1}{2}\left[\sin\left(\frac{\pi n}{16} + \frac{n\pi}{17}\right) + \sin\left(\frac{\pi n}{16} - \frac{n\pi}{17}\right)\right]$$

$$= \frac{1}{2}\cos\left(\frac{33\pi n}{272}\right) + \frac{1}{2}\cos\left(\frac{\pi n}{272}\right) + \frac{1}{2}\sin\left(\frac{33\pi n}{272}\right) +$$

$$\frac{1}{2}\sin\left(\frac{\pi n}{272}\right) \text{ .... (i)}$$

We know, for a DT signal, the signal is periodic if its fundamental frequency is a ratio of two integers.

Also, for $x[n] = x_1[n] + x_2[n]$ to be periodic both $x_1[n]$ and $x_2[n]$ must be periodic.

Comparing (i) with $x[n] = x_1[n] + x_2[n] + x_3[n] + x_4[n]$, we have

$$x_1[n] = \frac{1}{2}\cos\left(\frac{33\pi n}{272}\right)$$

$$x_2[n] = \frac{1}{2}\cos\left(\frac{\pi n}{272}\right)$$

$$x_3[n] = \frac{1}{2}\sin\left(\frac{33\pi n}{272}\right)$$

$$x_4[n] = \frac{1}{2}\sin\left(\frac{\pi n}{272}\right)$$

Case-I: $x_1[n] = \frac{1}{2}\cos\left(\frac{33\pi n}{272}\right)$

Comparing with $x[n] = \cos\omega_0 n$, we have

$$\omega_0 = \frac{33\pi}{272}$$

$$\text{or, } 2\pi f_0 = \frac{33\pi}{272}$$

$$\Rightarrow f_0 = \frac{33\pi}{272} \times \frac{1}{2\pi} = \frac{33}{544} = \frac{k}{N}$$

Case-II: $x_2[n] = \frac{1}{2}\cos\left(\frac{\pi n}{272}\right)$

Comparing with $x[n] = \cos\omega_0 n$, we have

$$\omega_0 = \frac{\pi}{272}$$

$$\text{or, } 2\pi f_0 = \frac{\pi}{272}$$

$$\Rightarrow f_0 = \frac{\pi}{272} \times \frac{1}{2\pi} = \frac{1}{544} = \frac{k}{N}$$

Case-III: $x_3[n] = \frac{1}{2}\sin\left(\frac{33\pi n}{272}\right)$

Comparing with $x[n] = \sin\omega_0 n$, we have

$$\omega_0 = \frac{33\pi}{272}$$

$$\Rightarrow f_0 = \frac{33}{544} = \frac{k}{N}$$

Case-IV: $x_4[n] = \frac{1}{2}\sin\left(\frac{\pi n}{272}\right)$

Comparing with $x[n] = \sin\omega_0 n$, we have

$$\omega_0 = \frac{\pi}{272}$$

$$\Rightarrow f_0 = \frac{1}{544} = \frac{k}{N}$$

Therefore, in all cases of $x_1[n]$, $x_2[n]$, $x_3[n]$ and $x_4[n]$, the fundamental frequency is a ratio of two integers. So, $x_1[n]$, $x_2[n]$, $x_3[n]$ and $x_4[n]$ are periodic. Hence, given signal $x[n] = e^{\frac{j\pi n}{16}}\cos\left(\frac{n\pi}{17}\right)$ is periodic.

Again,

Fundamental period of $x_1[n] = 544$

Fundamental period of $x_2[n] = 544$

Fundamental period of $x_3[n] = 544$

Fundamental period of $x_4[n] = 544$

Hence, fundamental period of $x[n]$ = LCM of period of $x_1[n]$, $x_2[n]$, $x_3[n]$ and $x_4[n] = 544$.

Example 1.9:

Check whether following signals are periodic or not. If yes, state their periodic time.

a. $x[n] = \sin(n\pi) + \cos(n\pi)$

b. $x[n] = \sin\left(\frac{3n\pi}{5}\right) + \cos\left(\frac{4n\pi}{7}\right)$ **[2080 Baishakh]**

Solution:

We know, for a DT signal, the signal is periodic if its fundamental frequency is a ratio of two integers.

Also, for $x[n] = x_1[n] + x_2[n]$ to be periodic, both $x_1[n]$ and $x_2[n]$ must be periodic.

a. Here, $x[n] = \sin(n\pi) + \cos(n\pi)$

Comparing with $x[n] = x_1[n] + x_2[n]$, we have

$x_1[n] = \sin(n\pi)$

$x_2[n] = \cos(n\pi)$

Case-I: $x_1[n] = \sin(n\pi)$

Comparing with $x[n] = \sin\omega_0 n$, we have

$\omega_0 = \pi$

or, $2\pi f_0 = \pi$

or, $f_0 = \frac{\pi}{2\pi} = \frac{1}{2} = \frac{k}{N}$

Since, the fundamental frequency is a ratio of two integers, $x_1[n]$ is periodic.

$\therefore$ Period of $x_1[n] = 2$

Case-II: $x_2[n] = \cos(n\pi)$

Comparing with $x[n] = \cos\omega_0 n$, we have

$\omega_0 = \pi$

or, $2\pi f_0 = \pi$

$\Rightarrow f_0 = \frac{\pi}{2\pi} = \frac{1}{2} = \frac{k}{N}$

Since, the fundamental frequency is a ratio of two integers, $x_2[n]$ is periodic.

$\therefore$ Period of $x_2[n] = 2$

Hence, given signal $x[n] = \sin(n\pi) + \cos(n\pi)$ is periodic since both $\sin(n\pi)$ and $\cos(n\pi)$ are periodic.

Period of $x[n]$ = LCM of period of $x_1[n]$ and $x_2[n]$

$= 2$

b. Here, $x[n] = \sin\left(\frac{3n\pi}{5}\right) + \cos\left(\frac{4n\pi}{7}\right)$

Comparing with $x[n] = x_1[n] + x_2[n]$, we have

$x_1[n] = \sin\left(\frac{3n\pi}{5}\right)$

$x_2[n] = \cos\left(\frac{4n\pi}{7}\right)$

Case-I: $x_1[n] = \sin\left(\frac{3n\pi}{5}\right)$

Comparing with $x[n] = \sin\omega_0 n$, we have

$\omega_0 = \frac{3\pi}{5}$

or, $2\pi f_0 = \frac{3\pi}{5}$

$\Rightarrow f_0 = \frac{3\pi}{5} \times \frac{1}{2\pi} = \frac{3}{10} = \frac{k}{N}$

Since the fundamental frequency is a ratio of two integers, $x_1[n]$ is periodic.

Period of $x_1[n] = 10$

Case-II: $x_2[n] = \cos\left(\frac{4n\pi}{7}\right)$

Comparing with $x[n] = \cos\omega_0 n$, we have

$\omega_0 = \frac{4\pi}{7}$

or, $2\pi f_0 = \frac{4\pi}{7}$

$\Rightarrow f_0 = \frac{4\pi}{7} \times \frac{1}{2\pi} = \frac{2}{7} = \frac{k}{N}$

Since the fundamental frequency is a ratio of two integers, $x_2[n]$ is periodic.

Period of $x_2[n] = 7$

Hence, the given signal $x[n] = \sin\left(\frac{3n\pi}{5}\right) + \cos\left(\frac{4n\pi}{7}\right)$ is periodic since both $\left(\frac{3n\pi}{5}\right)$ and $\cos\left(\frac{4n\pi}{7}\right)$ are periodic.

Period of $x[n]$ = LCM of period of $x_1[n]$ and $x_2[n]$

$= 70$

Example 1.10:

Determine whether the signal $x[n] = \cos\left(\frac{\pi n}{2}\right)\cos\left(\frac{\pi n}{4}\right)$ is periodic or non periodic and if it is periodic, find its fundamental period. [2078 Bhadra]

Solution:

Given signal is, $x[n] = \cos\left(\frac{\pi n}{2}\right)\cos\left(\frac{\pi n}{4}\right)$

or, $x[n] = \frac{1}{2} \times 2\cos\left(\frac{\pi n}{2}\right)\cos\left(\frac{\pi n}{4}\right)$

$= \frac{1}{2}\left[\cos\left(\frac{\pi n}{2} + \frac{\pi n}{4}\right) + \cos\left(\frac{\pi n}{2} - \frac{\pi n}{4}\right)\right]$

or, $x[n] = \frac{1}{2}\cos\left(\frac{3n\pi}{4}\right) + \frac{1}{2}\cos\left(\frac{\pi n}{4}\right)$ .... (ii)

We know, for a DT signal, the signal is periodic if its fundamental frequency is a ratio of two integers.

Also, for $x[n] = x_1[n] + x_2[n]$ to be periodic, both $x_1[n]$ and $x_2[n]$ must be periodic.

Comparing (i) with $x[n] = x_1[n] + x_2[n]$, we have

$x_1[n] = \frac{1}{2}\cos\left(\frac{3n\pi}{4}\right)$

$x_2[n] = \frac{1}{2}\cos\left(\frac{n\pi}{4}\right)$

Case-I: $x_1[n] = \frac{1}{2}\cos\left(\frac{3n\pi}{4}\right)$

Comparing with $x[n] = \cos\omega_0 n$, we have,

$\omega_0 = \frac{3\pi}{4}$

or, $2\pi f_0 = \frac{3\pi}{4}$

$$\Rightarrow f_0 = \frac{3\pi}{4} \times \frac{1}{2\pi} = \frac{3}{8}$$

Since, the fundamental frequency is a ratio of two integers, $x_1[n]$ is periodic.

Period of $x_1[n] = 8$

Case-II: $x_2[n] = \frac{1}{2}\cos\left(\frac{n\pi}{4}\right)$

Comparing with $x[n] = \cos(\omega_0 n)$, we have

$$\omega_0 = \frac{\pi}{4}$$

$$\text{or, } 2\pi f_0 = \frac{\pi}{4}$$

$$\Rightarrow f_0 = \frac{\pi}{4} \times \frac{1}{2\pi} = \frac{1}{8}$$

Since, the fundamental frequency is a ratio of two integers, $x_2[n]$ is periodic.

Period of $x_2[n] = 8$

Hence, the given signal $x[n] = \cos\left(\frac{\pi n}{4}\right)\cos\left(\frac{\pi n}{4}\right)$ is periodic.

Period of $x[n]$ = LCM of period of $x_1[n]$ and $x_2[n]$

$= 8$

Even and Odd Signal

- **Even signal (symmetric signal):** A real valued signal $x[n]$ is called *even signal* if $x[-n] = x[n]$. It is denoted by $x_e[n]$. Example: $x_e[n] = A\cos\omega_0 n$.

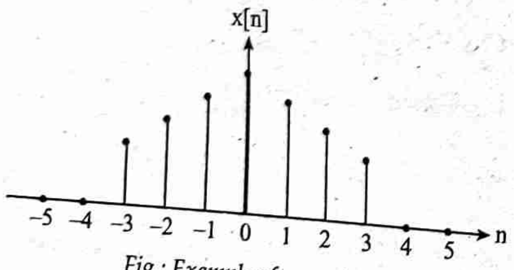


Fig.: Example of even signal

- **Odd signal (antisymmetric signal):** A real valued signal $x[n]$ is called *odd signal* if $x[-n] = -x[n]$. It is denoted by $x_o[n]$. Example: $x_o[n] = A\sin\omega_0 n$

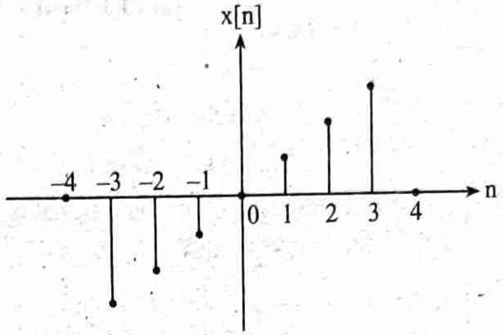


Fig.: Example of odd signal

Derivation of Even and Odd Parts of a Generic Signal

Let us consider a signal $x[n]$ such that it is a composition of even and odd parts.

$x[n] = x_e[n] + x_o[n]$ .... (i)

Substitute $n = -n$, we get

$x[-n] = x_e[-n] + x_o[-n]$

or, $x[-n] = x_e[n] - x_o[n]$ .... (ii)

∵ Even signal: $x_e[-n] = x_e[n]$

Odd signal: $x_o[-n] = -x_o[n]$

Adding equations (i) and (ii), we get

$x[n] + x[-n] = x_e[n] + x_o[n] + x_e[n] - x_o[n]$

or, $x[n] + x[-n] = 2x_e[n]$

$\therefore\ x_e[n] = \frac{x[n] + x[-n]}{2}$; even component

Subtracting equation (ii) from (i), we get

$x[n] - x[-n] = x_e[n] + x_o[n] - [x_e[n] - x_o[n]]$

or, $x[n] - x[-n] = x_e[n] + x_o[n] - x_e[n] + x_o[n]$

or, $x[n] - x[-n] = 2x_o[n]$

$\therefore\ x_o[n] = \frac{x[n] - x[-n]}{2}$; odd component

Example 1.11:

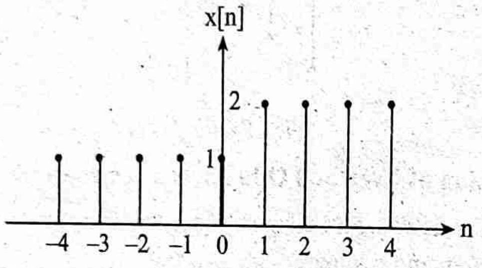
Find the even and odd part of signal x[n],

$$x[n] = \begin{cases} 1 \text{ for } -4 \le n \le 0 \\ 2 \text{ for } 1 \le n \le 4 \end{cases}$$ **[2071 Chaitra, 2070 Ashadh]**

Solution:

Given signal is: $x[n] = \begin{cases} 1 \text{ for } -4 \le n \le 0 \\ 2 \text{ for } 1 \le n \le 4 \end{cases}$

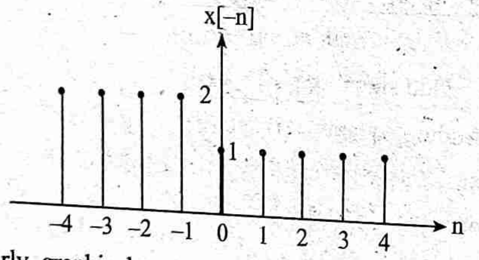
The graphical representation of given signal is



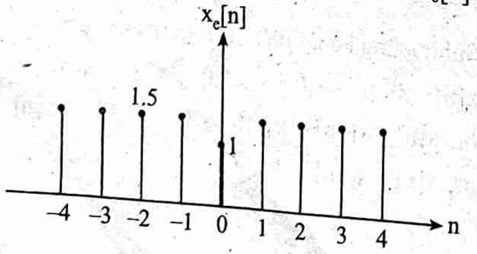
Even part of signal x[n], $x_e[n] = \frac{x[n] + x[-n]}{2}$

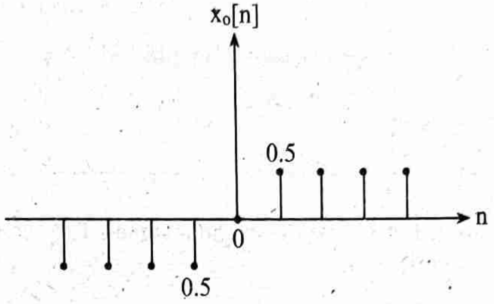
Odd part of signal x[n], $x_o[n] = \frac{x[n] - x[-n]}{2}$

Here, graphical representation of signal x[–n] is



Similarly, graphical representation of $x_e[n]$ and $x_o[n]$ are,





1.4 Transformation of Independent Variable

During the processing of a signal x[n], we perform various manipulations involving the independent variable n. These manipulations help for simplifying the signal processing. They can also be reversed when needed. After the manipulations, the signal's appearance looks changed, although the information of the new signal is the same as the original signal.

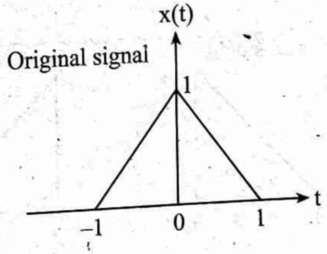
The most commonly used transformations are:

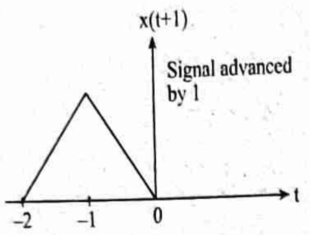
i. Shifting
ii. Scaling
iii. Folding or inversion

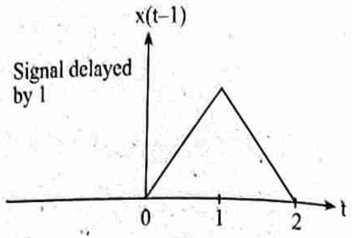
i. **Shifting:** Let x(t) be the original signal. The shifted version of x(t) is x(t + a) and x(t – a).

- **x(t + a):** Signal is advanced by a. We shift x(t) to left on time axis by a.
- **x(t – a):** Signal is delayed by a. We shift x(t) to right on time axis by a.

Example:



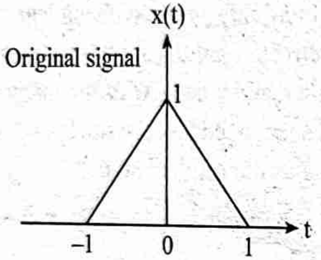


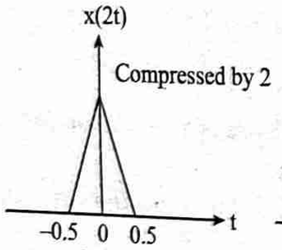


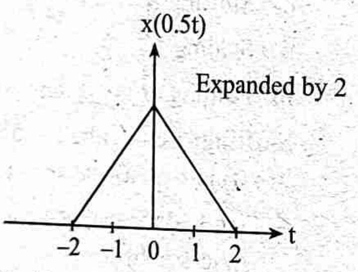
ii. **Scaling:** Let x(t) be the original signal. The scaled version of x(t) is x(at).

- In x(at), if a < 1, the signal is expanded by $\frac{1}{a}$.
- In x(at), if a > 1, the signal is compressed by a.

Example:

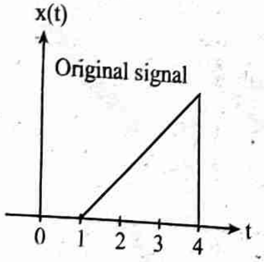


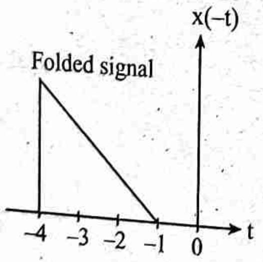




iii. **Inversion or folding:** Let x(t) be the original signal. The folded version of x(t) is x(−t).

Example:





Multitransformation

At times, we may be required to evolve a new signal where more than one transformation are there. In such cases, we proceed sequentially as follows:

i. Shifting
ii. Scaling
iii. Folding

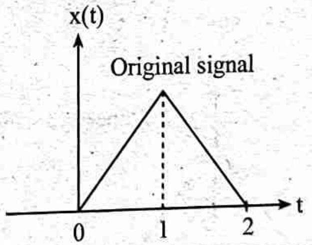
For example, to find $3x\left(\frac{-1}{2}t+1\right)$ from x(t), we follow the following steps:

i. Draw x(t).
ii. Advance x(t) by 1 to get x(t + 1).
iii. Scale x(t + 1) by $\frac{1}{2}$ to get $x\left(\frac{1}{2}t+1\right)$.
iv. Inverse $x\left(\frac{1}{2}t+1\right)$ to obtain $x\left(\frac{-1}{2}t+1\right)$.
v. Finally multiply $x\left(\frac{-1}{2}t+1\right)$ by 3 to obtain $3x\left(\frac{-1}{2}t+1\right)$.

Example 1.12:

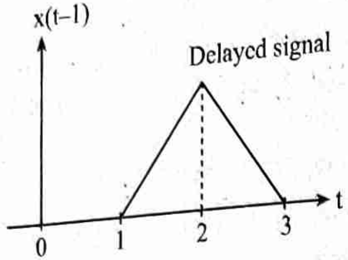
For given signal; find

$x(t-1), x(t+1), x(-t), x(1-t), x(2t+1), x\left(4-\frac{t}{2}\right)$.

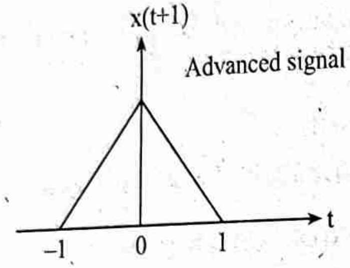


Solution:

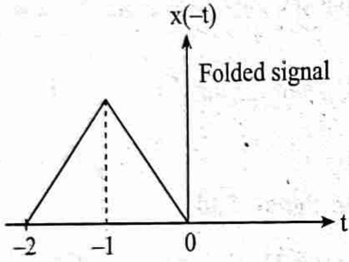
i. **x(t − 1)**



ii. $x(t + 1)$

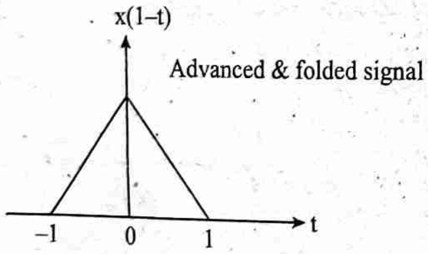


iii. $x(-t)$



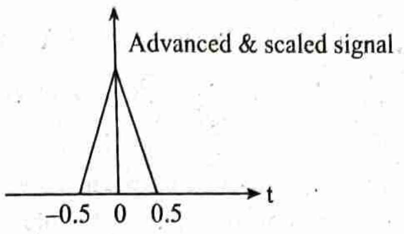
iv. $x(1 - t)$

$\Rightarrow x(t+1) \rightarrow x(-t+1)$



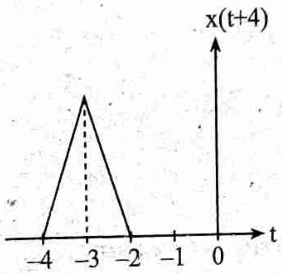
v. $x(2t + 1)$

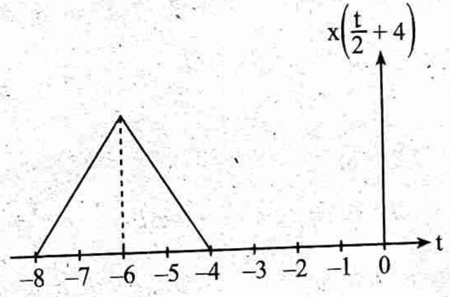
$\Rightarrow x(t+1) \rightarrow x(2t+1)$

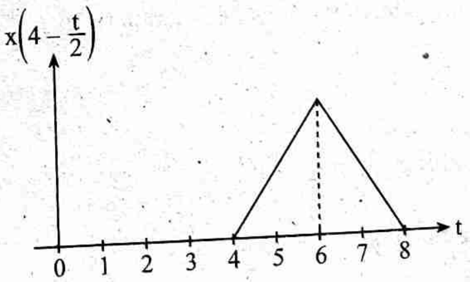


vi. $x\left(4 - \frac{t}{2}\right)$

$\Rightarrow x(t+4) \rightarrow x\left(\frac{t}{2}+4\right) \rightarrow x\left(\frac{-t}{2}+4\right)$







1.5 Discrete Time Fourier Series and Properties

Let us consider a DT periodic signal x[n] with periods N such that $x[n] = x[n + N]$.

A *Fourier series* is an expansion of a periodic signal in terms of sum of harmonically related complex exponentials. It is used for spectral analysis (analysis of magnitude and phase spectrum) of a periodic signal. It can be used for analysis of only periodic signals.

Synthesis equation:

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} \qquad ; \omega_0 = \frac{2\pi}{N}$$

Analysis equation:

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\omega_0 n} \qquad ; \omega_0 = \frac{2\pi}{N}$$

Derivation of Analysis Equation

Assume that synthesis equation is true,

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} \quad \text{......... (i)}$$

where $\omega_0 = \dfrac{2\pi}{N}$.

The synthesis equation synthesizes the signal x[n] from the coefficients a_k, k = 0, 1, 2, 3, ..., N – 1. The coefficients a_k are called Fourier coefficients or spectral coefficients of x[n].

Multiplying both sides of equation (i) by $e^{-jl\omega_0 n}$,

$$x[n]\, e^{-jl\omega_0 n} = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} \times e^{-jl\omega_0 n}$$

$$\text{or, } x[n]\, e^{-jl\omega_0 n} = \sum_{k=0}^{N-1} a_k e^{j(k-l)\,\omega_0 n}$$

Summing both sides, we get

$$\sum_{n=0}^{N-1} x[n]\, e^{-jl\omega_0 n} = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} a_k e^{j(k-l)\omega_0 n} \quad \text{....... (ii)}$$

We know,

$$\sum_{n=0}^{N-1} a^n = N \text{ for } a = 1$$

So,

$$\sum_{n=0}^{N-1} e^{jk\omega_0 n} = \sum_{n=0}^{N-1} (e^{jk\omega_0})^n = N \text{ for } k = 0$$

Hence,

$$\sum_{n=0}^{N-1} e^{j(k-l)\omega_0 n} = N \text{ for } k = l$$

Therefore at $k = l$, equation (ii) becomes

$$\sum_{n=0}^{N-1} x[n]e^{-jl\omega_0 n} = a_l N$$

$$\text{or, } a_l = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jl\omega_0 n}$$

In general,

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\omega_0 n}$$

which is the required analysis equation.

Properties of DT Fourier Series

i. **Linearity property:**

If $x[n] \xleftrightarrow{DTFS} a_k$

and, $y[n] \xleftrightarrow{DTFS} b_k$, then

$Ax[n] + By[n] \xleftrightarrow{DTFS} Aa_k + Bb_k$

ii. **Time reversal:**

If $x[n] \xleftrightarrow{DTFS} a_k$, then

$x[-n] \xleftrightarrow{DTFS} a_{-k}$

iii. **Time shifting:**

If $x[n] \xleftrightarrow{DTFS} a_k$, then

$$x[n-n_0] \xleftrightarrow{\text{DTFS}} a_k e^{-jk\omega_0 n_0}$$

iv. **Frequency shifting:**

If $x[n] \xleftrightarrow{\text{DTFS}} a_k$, then

$$e^{jm\omega_0 n} x[n] \xleftrightarrow{\text{DTFS}} a_{k-m}$$

v. **Multiplication:**

If $x[n] \xleftrightarrow{\text{DTFS}} a_k$

and, $y[n] \xleftrightarrow{\text{DTFS}} b_k$, then

$$x[n]y[n] \xleftrightarrow{\text{DTFS}} \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

vi. **Conjugation:**

If $x[n] \xleftrightarrow{\text{DTFS}} a_k$, then

$$x^*[n] \xleftrightarrow{\text{DTFS}} a^*_{-k}$$

Parseval's Theorem for DT Periodic Signal

If $x[n] \xleftrightarrow{\text{DTFS}} a_k$, where x[n] is periodic with period N, then

$$P_{avg} = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |a_k|^2$$

Proof:

If x[n] is periodic with period N, then, we have

$$x[n] = \sum_{k=0}^{N-1} a_k\, e^{jk\omega_0 n}$$

and, $a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\omega_0 n}$

Now,

$$P_{avg} = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, x^*[n]$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x[n] \sum_{k=0}^{N-1} a^*_k\, e^{-jk\omega_0 n}$$

$$= \sum_{k=0}^{N-1} a^*_k \times \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\omega_0 n}$$

$$= \sum_{k=0}^{N-1} a^*_k\, a_k$$

$$= \sum_{k=0}^{N-1} |a_k|^2 \text{ proved.}$$

Example 1.13:

Find the coefficients of $x[n] = \cos\left(\frac{\pi n}{3}\right)$.

Solution:

Given, $x[n] = \cos\left(\frac{\pi n}{3}\right)$

Comparing with $x[n] = \cos\omega_0 n$, we have

$$\omega_0 = \frac{\pi}{3}$$

Using Euler's formula,

$$x[n] = \cos\left(\frac{\pi n}{3}\right) = \frac{e^{j\pi n/3} + e^{-j\pi n/3}}{2}$$

$$\text{or, } x[n] = \frac{1}{2}e^{j\pi n/3} + \frac{1}{2}e^{-j\pi n/3}$$

$$\text{or, } x[n] = \frac{1}{2}e^{j\pi n/3} + \frac{1}{2}e^{j(-\pi/3 + 2\pi)n}$$

$$\text{or, } x[n] = \frac{1}{2}e^{j\pi n/3} + \frac{1}{2}e^{j5\pi n/3}$$

Comparing with synthesis equation,

$x[n] = \sum_{k=0}^{N-1} a_k\, e^{jk\omega_0 n}$, we have

$$a_1 = \frac{1}{2} \text{ and } a_5 = \frac{1}{2}$$

Also,

$$a_0 = a_2 = a_3 = a_4 = 0$$

Example 1.14:

Find the coefficients of $x[n] = \cos(\sqrt{2}\,\pi n)$.

Solution:

Given, $x[n] = \cos(\sqrt{2}\,\pi n)$

Comparing with $x[n] = \cos\omega_0 n$, we have

$$\omega_0 = \sqrt{2}\,\pi$$

$$\text{or, } 2\pi f_0 = \sqrt{2}\,\pi$$

$$\Rightarrow f_0 = \frac{\sqrt{2}\pi}{2\pi} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

Since, the fundamental frequency is not a ratio of two integers, the given signal is not periodic. Therefore, it doesn't have Fourier coefficients.

Example 1.15:

Find the coefficients of $x[n] = \{1, 1, 0, 0\}$.

Solution:

Given, $x[n] = \{1, 1, 0, 0\}$

Here, $N = 4$

$$\omega_0 = \frac{2\pi}{N} = \frac{\pi}{2}$$

$x(0) = 1, x(1) = 1, x(2) = 0, x(3) = 0$

Analysis equation is

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\omega_0 n} \qquad ; k = 0, 1, 2, 3$$

To calculate a_0,

$$a_0 = \frac{1}{4}\sum_{n=0}^{3} x[n]\, e^{0}$$

$$= \frac{1}{4}\{x(0) + x(1) + x(2) + x(3)\}$$

$$\therefore\ a_0 = \frac{1+1+0+0}{4} = \frac{2}{4} = \frac{1}{2}$$

To calculate a_1,

$$a_1 = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\omega_0 n}$$

$$= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\pi n/2}$$

$$= \frac{1}{4}\{x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi} + x(3)e^{-j3\pi/2}\}$$

$$= \frac{1}{4}(1 + 1\times e^{-j\pi/2} + 0 + 0)$$

$$= \frac{1}{4}[1 + \cos(\pi/2) - j\sin(\pi/2)]$$

$$= \frac{1}{4}(1 + 0 - j)$$

$$\therefore\ a_1 = \frac{1-j}{4}$$

To calculate a_2,

$$a_2 = \frac{1}{4}\sum_{n=0}^{3} x[n]\, e^{-2j\omega_0 n}$$

$$= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\pi n}$$

$$= \frac{1}{4}\{x(0)e^{0} + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}\}$$

$$= \frac{1}{4}(1 + 1\times e^{-j\pi} + 0 + 0)$$

$$= \frac{1}{4}[1 + \cos(\pi) - j\sin(\pi)]$$

$$= \frac{1}{4}(1 - 1 - 0)$$

$$\therefore\ a_2 = 0$$

To calculate a_3,

$$a_3 = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j3\omega_0 n}$$

$$= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j3\pi n/2}$$

$$= \frac{1}{4}\{x(0)e^{0} + x(1)e^{-j3\pi/2} + x(2)e^{-j3\pi} + x(3)e^{-j9\pi/2}\}$$

$$= \frac{1}{4}(1 + 1\times e^{-j3\pi/2} + 0 + 0)$$

$$= \frac{1}{4}\{1 + \cos(3\pi/2) - j\sin(3\pi/2)\}$$

$$= \frac{1}{4}\{1 + 0 - j(-1)\}$$

$$\therefore\ a_3 = \frac{1+j}{4}$$

Hence,

$$a_0 = \frac{1}{2};\ |a_0| = \frac{1}{2},\ \angle a_0 = 0°$$

$$a_1 = \frac{1}{4}(1-j);\ |a_1| = \frac{1}{4}\sqrt{(1)^2 + (-1)^2} = 0.3535,$$

$$\angle a_1 = \tan\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$$

$$a_2 = 0;\ |a_2| = 0,\ \angle a_2 = 0$$

$$a_3 = \frac{1}{4}(1+j);\ |a_3| = \frac{1}{4}\sqrt{(1)^2 + (1)^2} = 0.3535$$

$$\angle a_3 = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

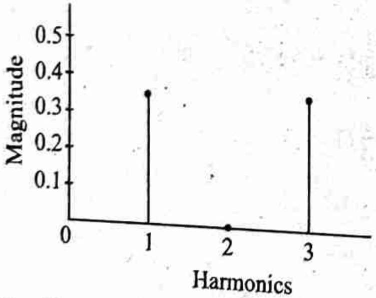


Fig.: Plot of magnitude at each harmonics

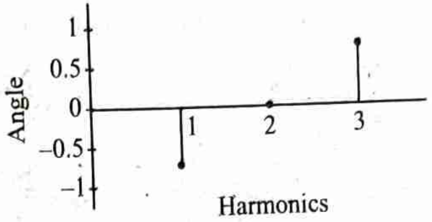


Fig.: Plot of phase angle at each harmonics

1.6 Discrete Time Fourier Transform and Properties

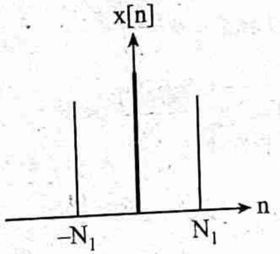
Fourier transform is a mathematical operation for converting a signal from time domain into its frequency domain.

Let us consider a DT signal x[n]. Its Fourier transform is given by $X(e^{j\omega})$ and is defined as

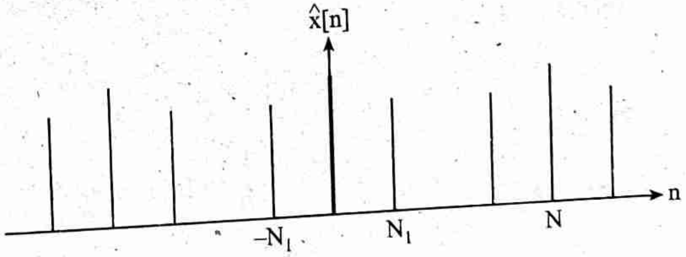
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

Derivation of Fourier Transform from Fourier Series

Let us consider a DT aperiodic signal x[n]



Suppose $\hat{x}[n]$ be the periodic signal formed by repeating x[n] over the period N.



Since $\hat{x}[n]$ is periodic, we can apply Fourier series. So,

$$\hat{x}[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad ; \omega_0 = \frac{2\pi}{N}$$

$$\text{and, } a_k = \frac{1}{N} \sum_{n=<N>} \hat{x}[n] e^{-jk\omega_0 n} \qquad ; \omega_0 = \frac{2\pi}{N}$$

In the range $\frac{-N}{2}$ to $\frac{N}{2} - 1$, $\hat{x}[n]$ tends to x[n].

$$\text{or, } Na_k = \sum_{-N/2}^{N/2-1} x[n] e^{-jk\omega_0 n}$$

Let us assume $Na_k = X(e^{j\omega})$ and $k\omega_0 = \omega$ for $N \to \infty$ then

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

This is the equation of *Discrete Time Fourier Transform (DTFT).*

We have,

$$Na_k = X(e^{j\omega}) \text{ and } a_k = \frac{X(e^{j\omega})}{N}$$

$$\text{Hence, } \hat{x}[n] = \sum_{k=<N>} \frac{X(e^{j\omega})}{N} e^{jk\omega_0 n}$$

$$\text{or, } \hat{x}[n] = \sum_{k=<N>} X(e^{j\omega}) e^{jk\omega_0 n} \frac{\omega_0}{2\pi} \qquad \left[\because \omega_0 = \frac{2\pi}{N}\right]$$

$$\text{or, } \hat{x}[n] = \frac{1}{2\pi} \sum_{k=<N>} X(e^{jw}) e^{jk\omega_0 n} \omega_0$$

Therefore, as $N \to \infty$,

$\hat{x}[n] \to x[n]$

$\omega_0 \to 0$

$\Sigma \to \int$

Let $\omega = k\omega_0$, then

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Example 1.16:

Find Fourier Transform: x[n] = δ[n]

Solution:

Given, x[n] = δ[n]

We know,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$\delta[n] = \begin{cases} 1 & \text{at } n = 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{So, } X(e^{j\omega}) = \sum_{n=0} x[n] e^{-j\omega n}$$

$$= x(0)\, e^0$$

$$= 1 \times 1$$

$$\therefore X(e^{j\omega}) = 1$$

Example 1.17:

Find Fourier Transform: $x[n] = a^n u[n]$; $|a| < 1$.

Solution:

Given, $x[n] = a^n u[n]$

We have,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n}$$

$$\text{We have, } u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

So, $X(e^{j\omega}) = \sum\limits_{n=0}^{\infty} a^n e^{-j\omega n}$

$$= \sum_{n=0}^{\infty} (a\, e^{-j\omega})^n$$

$$= \frac{1}{1 - ae^{-j\omega}} \left[\because \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \right]$$

Example 1.18:

Find the Fourier transform of $x[n] = \begin{cases} 1, & -N_1 \le n \le N_1 \\ 0, & \text{otherwise} \end{cases}$

Solution:

Given, $x[n] = \begin{cases} 1, & -N_1 \le n \le N_1 \\ 0, & \text{otherwise} \end{cases}$

We know,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$= \sum_{n=-N_1}^{N_1} x[n]e^{-j\omega n}$$

$$= \sum_{n=-N_1}^{N_1} e^{-j\omega n}$$

Let $m = n + N_1$, then

$$X(e^{j\omega}) = \sum_{m=0}^{2N_1} e^{-j\omega(m-N_1)}$$

$$= \sum_{m=0}^{2N_1} e^{-j\omega m} e^{j\omega N_1}$$

$$= e^{j\omega N_1} \sum_{m=0}^{2N_1} e^{-j\omega m}$$

$$= e^{j\omega N_1} \sum_{m=0}^{2N_1} (e^{-j\omega})^m$$

We have, $\sum\limits_{m=0}^{N} a^n = \dfrac{1 - a^{N+1}}{1-a}$. So,

$$X(e^{j\omega}) = e^{j\omega N_1} \times \frac{1 - (e^{-j\omega})^{2N_1+1}}{1 - e^{-j\omega}}$$

$$= e^{j\omega N_1} \times \frac{1 - e^{-j\omega(2N_1+1)}}{1 - e^{-j\omega}}$$

We know, $\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$. So,

$$X(e^{j\omega}) = e^{j\omega N_1} \frac{\left\{ \dfrac{e^{\frac{-j\omega(2N_1+1)}{2}} \left[e^{\frac{j\omega(2N_1+1)}{2}} - e^{\frac{-j\omega(2N_1+1)}{2}} \right]}{2j} \right\}}{\left\{ \dfrac{e^{\frac{-j\omega}{2}} \left[e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}} \right]}{2j} \right\}}$$

$$\text{or, } X(e^{j\omega}) = e^{j\omega N_1} \times \frac{e^{\frac{-j\omega(2N_1+1)}{2}}}{e^{\frac{-j\omega}{2}}} \times \frac{\sin\left\{\frac{\omega}{2}(2N_1+1)\right\}}{\sin\left(\frac{\omega}{2}\right)}$$

$$\text{or, } X(e^{j\omega}) = e^{j\omega N_1 - \frac{j\omega(2N_1+1)}{2} + \frac{j\omega}{2}} \times \frac{\sin\left\{\frac{\omega}{2}(2N_1+1)\right\}}{\sin\left(\frac{\omega}{2}\right)}$$

$$= e^0 \times \frac{\sin\left\{\frac{\omega}{2}(2N_1+1)\right\}}{\sin\left(\frac{\omega}{2}\right)}$$

$$\therefore\ X(e^{j\omega}) = \frac{\sin\left\{\frac{\omega}{2}(2N_1+1)\right\}}{\sin\left(\frac{\omega}{2}\right)}$$

Properties of Discrete Time Fourier Transform (DTFT)

i. Periodicity:

DTFT of any signal is periodic about 2π.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$\therefore\ X(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn(\omega+2\pi)}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega}\times e^{-j2\pi n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$[\because e^{-j2\pi n} = \cos(2\pi n) - j\sin(2\pi n) = 1 - 0 = 1]$$

$$= X(e^{j\omega})$$

$\therefore$ $X(e^{j\omega})$ is periodic in ω with period 2π.

ii. **Linearity:**

If $x[n] \xleftrightarrow{FT} X(e^{j\omega})$

and, $y[n] \xleftrightarrow{FT} Y(e^{j\omega})$, then

$ax[n] + by[n] \xleftrightarrow{FT} aX(e^{j\omega}) + bY(e^{j\omega})$

iii. **Time shifting:**

If $x[n] \xleftrightarrow{FT} X(e^{j\omega})$, then

$x[n - n_0] \xleftrightarrow{FT} e^{-j\omega n_0}X(e^{j\omega})$

iv. **Frequency shifting:**

If $x[n] \xleftrightarrow{FT} X(e^{j\omega})$, then

$e^{j\omega_0 n}x[n] \xleftrightarrow{FT} X(e^{j(\omega-\omega_0)})$

v. **Conjugation:**

If $x[n] \xleftrightarrow{FT} X(e^{j\omega})$, then

$x^*[n] \xleftrightarrow{FT} X^*(e^{-j\omega})$

vi. **Time reversal:**

If $x[n] \xleftrightarrow{FT} X(e^{j\omega})$, then

$x[-n] \xleftrightarrow{FT} X(e^{-j\omega})$

vii. **Convolution:**

If $x[n] \xleftrightarrow{FT} X(e^{j\omega})$

and, $y[n] \xleftrightarrow{FT} Y(e^{j\omega})$, then

$x[n] * y[n] \xleftrightarrow{FT} X(e^{j\omega})\,Y(e^{j\omega})$

Proof:

$$FT[x[n] * y[n]] = \sum_{n=-\infty}^{\infty} [x[n] * y[n]]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k)y(n-k)\right] e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} y(n-k)\, e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} y(n-k)\, e^{-j\omega(n-k)}\, e^{-j\omega k}$$

$$= \sum_{k=-\infty}^{\infty} x(k)e^{-j\omega k} \sum_{n=-\infty}^{\infty} y(n-k)e^{-j\omega(n-k)}$$

$$= X(e^{j\omega})\,Y(e^{j\omega})$$

Differences Between Fourier Series and Fourier Transform

Fourier Series	Fourier Transform
i. Fourier series is an expansion of a periodic signal in terms of sum of harmonically related complex exponentials.	i. Fourier transform is a mathematical operation for converting a signal from time domain into frequency domain.
ii. It can be applied to periodic signals only.	ii. It can be applied to periodic as well as aperiodic signals.
iii. It is used for spectral analysis of periodic signal.	iii. It is used to solve differential equation.
iv. For a DT signal x[n], its Fourier series is given by $x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n}$	iv. For a DT signal x[n], its Fourier transform is given by $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

Relation between Different Signals

- **Relation between unit impulse and unit step signal:**

  $\int \delta(t)dt = u(t)$

  $\frac{du(t)}{dt} = \delta(t)$

- **Relation between unit step and unit ramp signal:**

  $\int u(t)dt = r(t)$

  $\frac{dr(t)}{dt} = u(t)$

- **Relation between unit impulse and unit ramp signal:**

  $\iint \delta(t)dt = r(t)$

  $\frac{d^2r(t)}{dt^2} = \delta(t)$

Summary:

$$\delta(t) \xrightarrow{\text{integrate}} u(t) \xrightarrow{\text{integrate}} r(t)$$

$$r(t) \xrightarrow{\text{differentiate}} u(t) \xrightarrow{\text{differentiate}} \delta(t)$$

1.7 Discrete Time System and Properties

System

A *system* is an entity or processing device which acts on one or more inputs or excitations and produces one or more responses or outputs. A typical block diagram for a system is shown below:

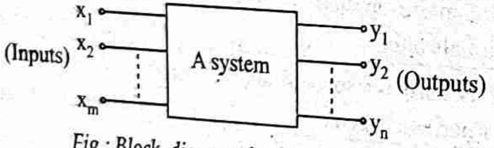


Fig.: Block, diagram for a typical system

Output of a system depends upon the transfer function.

Discrete Time (DT) System

A system is said to be a *discrete time system* if the associated signals are discrete time in nature. In DT system, both input and output signals are discrete in time. Example: Semiconductor memories, microprocessor, accumulator, etc.

Properties of System

1. **Static and dynamic system:**

   A system in which the output at any given time depends only upon the input at the same time and does not depend on past or future values of the input signal is called *static system*. It is also called *memoryless system*.

   Examples: $y[n] = 2x[n] - x^2[n]$, $y[n+2] = 5x[n+2]$

   A system in which the output at any given time depends on past or future value of the input signal is called *dynamic system*. It is also called *system with memory*.

   Examples: $y[n] = 2x[n-1] - x^2[n]$, $y[n] = 5x[n] + 3x[n+2]$

2. **Invertible and non-invertible system:**

   In *invertible system*, there exists one-to-one relationship between input and output signal. An inverse system exists in invertible system such that we can retrieve input from the output.

   Example: $y[n] = x[n]$

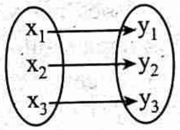


Fig.: Invertible system

   In *non-invertible system*, there exists many-to-one relationship between input and output signal. We cannot always retrieve input from the output. A single output can lead to multiple inputs.

   Example: $y[n] = x^2[n]$

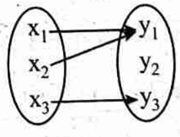


Fig.: Non-invertible system

3. **Causal and non-causal system:**

A system is *causal* if the output of the system at any time depends on only the present value or the past value of the input. A memoryless system has no delay or advances, and is, therefore, always a causal system. But a causal system is not always memory less, since, it can contain delays.

Examples: $y[n] = x[n] + x[n-1]$, $y[n] = \sum_{k=-\infty}^{n} x[k]$

A system is *non-causal* if the output of the system at any time depends on the future value of the input signal.

Example: $y[n] = x[n] + x[n+1]$

4. **Stable and unstable systems:** The stability of a system is defined in terms of BIBO (Bounded Input Bounded Output). If the input x[n] is finite, then output y[n] must be finite for BIBO stability.

Example: $y[n] = x^2[n]$ is BIBO stable

$$y[n] = \frac{1}{x[n]-1} \text{ is not BIBO stable}$$

5. **Time-invariant and time-variant systems:** *Time-invariant* means that system does not vary as time passes. In such system, the shift in input results in corresponding shift in the output. In discrete system, the system is time-invariant if it satisfies the conditions:

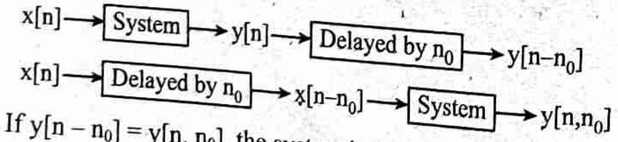
If response $[x[n]] = y[n]$

Then, response $[x[n-n_0]] = y[n-n_0]$ ...... (i)

That is, if the input is delayed by n_0, the output is also delayed by the same amount n_0.

The system is time-variant if the equation (i) is not satisfied.

To check time invariance,



If $y[n-n_0] = y[n, n_0]$, the system is time-invariant.

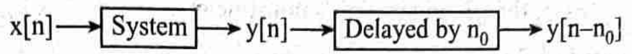
Example 1.19:

Check the time invariance for y[n] = x[2n].

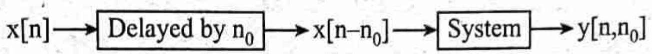
Solution:

Given, $y[n] = x[2n]$

Here,



$$y[n-n_0] = x[2n-n_0]$$



$$y[n, n_0] = x[2(n-n_0)]$$

Since $y[n-n_0] \neq y[n, n_0]$, the given signal is not time-invariant.

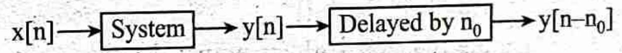
Example 1.20:

Check the time invariance for y[n] = sin[x(n)].

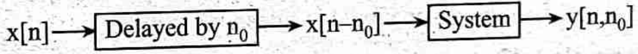
Solution:

Given, $y[n] = \sin[x(n)]$

Here,



$$y[n-n_0] = \sin[x(n-n_0)]$$



$$y[n, n_0] = \sin[x(n-n_0)]$$

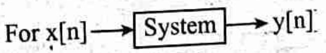
Since $y[n-n_0] = y[n, n_0]$, the given signal is time-invariant.

6. **Linear and non-linear systems:**

A system is said to be *linear* if it follows superposition and scaling property.

A system is said to be *non-linear* if it does not follow superposition and scaling property.

To check linearity:



$$x_1[n] \rightarrow \boxed{\text{System}} \rightarrow y_1[n]$$

$$x_2[n] \rightarrow \boxed{\text{System}} \rightarrow y_2[n]$$

$$\text{If } x_3[n]=ax_1[n]+bx_2[n] \rightarrow \boxed{\text{System}} \rightarrow y_3[n]=ay_1[n]+by_2[n]$$

then, the given system is linear.

Else, the given system is non-linear.

Example 1.21:

Check linearity: y[n] = nx[n].

Solution:

Given, $y[n] = nx[n]$

Here,

$$x[n] \rightarrow \boxed{\text{System}} \rightarrow y[n] = nx[n]$$

$$x_1[n] \rightarrow \boxed{\text{System}} \rightarrow y_1[n] = nx_1[n]$$

$$x_2[n] \rightarrow \boxed{\text{System}} \rightarrow y_2[n] = nx_2[n]$$

$$x_3[n] = ax_1[n] + bx_2[n] \rightarrow \boxed{\text{System}} \rightarrow y_3[n] = nx_3[n]$$

$$\begin{aligned} y_3[n] &= nx_3[n] \\ &= n\{ax_1[n] + bx_2[n]\} \\ &= anx_1[n] + bnx_2[n] \quad = ay_1[n] + by_2[n] \end{aligned}$$

Since, $y_3[n] = ay_1[n] + by_2[n]$ for input $x_3[n] = ax_1[n] + bx_2[n]$, the given signal is linear.

Example 1.22:

Check linearity: $y[n] = x^2[n]$. **[2074 Chaitra]**

Solution:

Given, $y[n] = x^2[n]$

Here,

$$x[n] \rightarrow \boxed{\text{System}} \rightarrow y[n] = x^2[n]$$

$$x_1[n] \rightarrow \boxed{\text{System}} \rightarrow y_1[n] = x_1^2[n]$$

$$x_2[n] \rightarrow \boxed{\text{System}} \rightarrow y_2[n] = x_2^2[n]$$

$$x_3[n] = ax_1[n] + bx_2[n] \rightarrow \boxed{\text{System}} \rightarrow y_3[n] = x_3^2[n]$$

$$\begin{aligned} y_3[n] &= x_3^2[n] \\ &= (ax_1[n] + bx_2[n])^2 \\ &= a^2x_1^2[n] + b^2x_2^2[n] + 2abx_1[n]x_2[n] \\ &= a^2y_1[n] + b^2y_2[n] + 2abx_1[n]x_2[n] \end{aligned}$$

Since $y_3[n] \neq ay_1[n] + by_2[n]$ for input $x_3[n] = ax_1[n]+ bx_2[n]$, the given signal is non-linear.

Example 1.23:

Check whether the following system is memoryless, causal, stable, linear and time-invariant or not.

$y[n] = \sin[x(n + 2)]$

Solution:

Given system is, $y[n] = \sin[x(n + 2)]$

Here,

$$x[n] \rightarrow \boxed{\text{System}} \rightarrow y[n] = \sin[x(n+2)]$$

i. The given system is dynamic i.e., not memoryless or system with memory since the output y[n] of the system depends on future value of the input signal, x[n+2].

ii. The given system is non-causal since the output y[n] of the system depends on future value of the input signal, x[n+2].

iii. The given system is stable since for finite value of input x[n], the value of output y[n] will be finite and always lies in the range –1 to 1.

iv. Linearity:

$$x_1[n] \rightarrow \boxed{\text{System}} \rightarrow y_1[n] = \sin[x_1(n+2)]$$

$$x_2[n] \rightarrow \boxed{\text{System}} \rightarrow y_2[n] = \sin[x_2(n+2)]$$

$$x_3[n] = ax_1[n] + bx_2[n] \rightarrow \boxed{\text{System}} \rightarrow y_3[n] = \sin[x_3(n+2)]$$

$$\begin{aligned} y_3[n] &= \sin[x_3(n + 2)] \\ &= \sin[ax_1(n + 2) + bx_2(n + 2)] \\ &= \sin[ax_1(n + 2)] \cos[bx_2(n + 2)] + \cos[ax_1(n + 2)] \\ &\quad \sin[bx_2(n + 2)] \end{aligned}$$

Since $y_3[n] \neq ay_1[n] + by_2[n]$ for input $x_3[n] = ax_1[n] + bx_2[n]$, the given system is non-linear.

v. **Time-invariance:**

$$x[n] \rightarrow \boxed{\text{System}} \rightarrow y[n] \rightarrow \boxed{\text{Delayed by } n_0} \rightarrow y[n-n_0]$$

$$y[n-n_0] = \sin[x(n+2-n_0)]$$

$$x[n] \rightarrow \boxed{\text{Delayed by } n_0} \rightarrow x[n-n_0] \rightarrow \boxed{\text{System}} \rightarrow y[n,n_0]$$

$$y[n, n_0] = \sin[x(n-n_0+2)]$$

Since $y[n-n_0] = y[n, n_0]$, the given system is time-invariant.

1.8 Linear Time-Invariant (LTI) System, Convolution Sum, Properties of LTI System

LTI System

A system which is both linear as well as time-invariant is termed as *linear-time invariant system (LTI)*. Most of the practical systems are LTI systems. The LTI system is characterized by the impulse response of system i.e., h[n]. The *impulse response* of a system is defined as its output when its input is impulse signal.

$$\delta[n] \rightarrow \boxed{\text{LTI System}} \rightarrow h[n]$$

$$\delta[n-k] \rightarrow \boxed{\text{LTI System}} \rightarrow h[n-k]$$

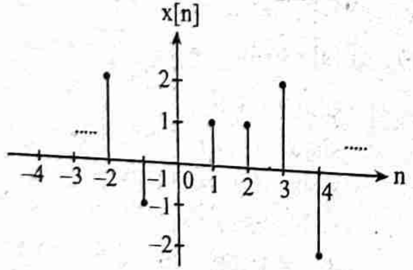
$$A\delta[n-k] \rightarrow \boxed{\text{LTI System}} \rightarrow Ah[n-k]$$

The output of Discrete Time Linear Time-Invariant System (DT LTI system) is calculated by convolution sum.

$$y[n] = x[n] * h[n]$$

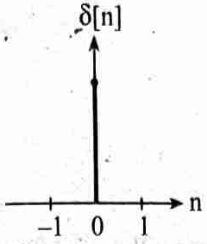
Convolution Sum

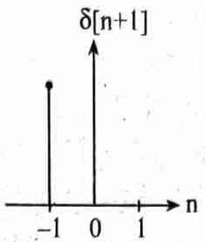
Let us consider an arbitrary DT signal x[n].

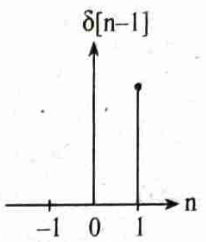


Here, x[n] = ... + x[–2] + x[–1] + x[0] + x[1] + x[2] + ...

We know,







So, we can write,

$$x[n] = ... + x[-2]\delta[n+2]+x[-1]\delta[n+1]+x[0]\delta[n]+x[1]\delta[n-1] + x[2]\,\delta[n-2] + ...$$

$$\text{or, } x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

Since,

$$x[n] \rightarrow \boxed{\text{System}} \rightarrow y[n]$$

$$\text{so, } y[n] = T\{x[n]\}$$

$$= T\left\{\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]\right\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\,T\{\delta[n-k]\}$$

$$\therefore\ y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

where h[n – k] is impulse response.

This is the convolution sum where x[n] is the input sequence and h[n] is the impulse response.

Example 1.24:

Find the convolution between two signals:

$$x_1[n] = \begin{cases} 1; & -1 \le n \le 1 \\ 0; & \text{otherwise} \end{cases}$$

$$\text{and, } x_2[n] = \begin{cases} 1; & -1 \le n \le 2 \\ 0; & \text{otherwise} \end{cases}$$

Solution:

Given signals are

$$x_1[n] = \begin{cases} 1; & -1 \le n \le 1 \\ 0; & \text{otherwise} \end{cases}$$

$$x_2[n] = \begin{cases} 1; & -1 \le n \le 2 \\ 0; & \text{otherwise} \end{cases}$$

Here,

$y[n] = x_1[n]*x_2[n]$ or $x_2[n]*x_1[n]$ [$\because$ Commutative property]

> **Note:**
>
> - We can use both $y[n] = x_1[n] * x_2[n]$
>
> $$= \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]$$
>
> and $y[n] = x_2[n] * x_1[n]$
>
> $$= \sum_{k=-\infty}^{\infty} x_2[k]\, x_1[n-k]$$
>
> - To simplify calculation, we take the first term such that it has smaller number of terms i.e. range is small. In this example, range of $x_1[n]$ is from –1 to 1 (3 terms) and range of $x_2[n]$ is from –1 to 2 (4 terms), so we take $x_1[n]$ as the first term.

$y[n] = x_1[n] * x_2[n]$

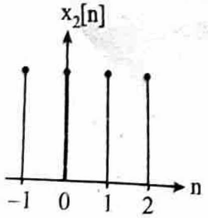
$$= \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]$$

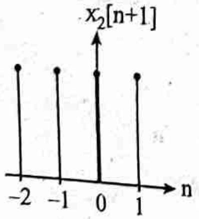
$$= \sum_{k=-1}^{1} x_1[k]\, x_2[n-k]$$

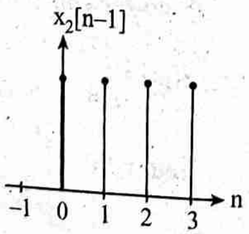
$$= x_1[-1]\, x_2[n+1] + x[0]\, x_2[n] + x_1[1]\, x_2[n-1]$$

$$= x_2[n+1] + x_2[n] + x_2[n-1]$$

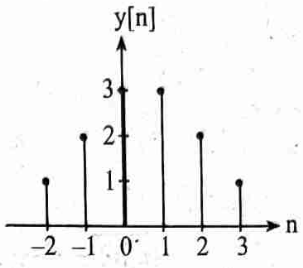
In graphical representation,



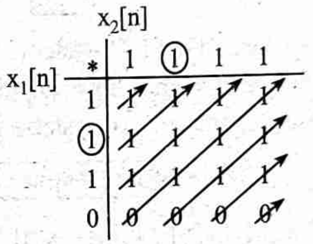




Therefore,



$$y[n] = \{1, 2, \underset{\uparrow}{3}, 3, 2, 1\}$$

> **Note:** To check answers:
>
> 
>
> | * | 1 | ① | 1 | 1 |
> |---|---|---|---|---|
> | 1 | 1 | 1 | 1 | 1 |
> | ① | 1 | 1 | 1 | 1 |
> | 1 | 1 | 1 | 1 | 1 |
> | 0 | 0 | 0 | 0 | 0 |
>
>
>
> Add diagonally,
>
> $$y[n] = \{1, 2, \underset{\uparrow}{3}, 3, 2, 1\}$$
>
> Zero at marked point intersection diagonal.

Properties of LTI System

1. Commutative property:

$x[n] * h[n] = h[n] * x[n]$

Proof:

Let, $y[n] = x[n] * h[n]$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Suppose $n - k = p$, then $k = n - p$. So,

$$y[n] = \sum_{p=\infty}^{-\infty} x[n-p]\, h[p]$$

$$= \sum_{p=-\infty}^{\infty} h[p]\, x[n-p]$$

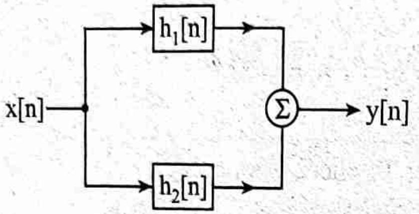
We can assume any variable in place of p.

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

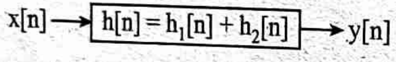
$$= h[n] * x[n]$$

Hence, $x[n] * h[n] = h[n] * x[n]$

2. **Distributive property:**

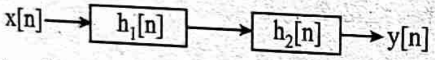


It states that "Two LTI systems in parallel can be replaced by a single system whose impulse response is the sum of impulse response of two systems in parallel.

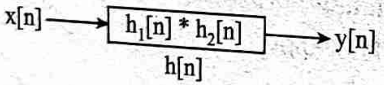


$x[n] * h_1[n] + x[n] * h_2[n] = x[n] * [h_1[n] + h_2[n]]$

3. **Associative property:**



It states that, "Two LTI system in series can be replaced by a single system whose impulse response is the convolution of impulse response of two systems in series.



4. **Causality:**

A system is said to be casual if the output of the system at any time depends on only the present value or the past value of the input.

A LTI system is said to be *causal* if $h[n] = 0$ for $n < 0$. In such system,

$$y[n] = \sum_{k=-\infty}^{n} x[k]\, h[n-k]$$

A LTI system is said to be *anti-causal* if h[n]=0 for $n > 0$. Also if $x[n] = 0$ for $n > 0$, the sequence x[n] is anti-causal.

5. **Stability:**

A DT LTI system is stable if its impulse response is absolutely summable i.e.,

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

6. **Memoryless:**

At LTI system is said to be *memoryless* if

$h[n] \neq 0$ for $n \neq 0$

$= 0$ otherwise

7. **Invertibility:**

For a LTI system with impulse response $h_1[n]$ to be *invertible*, there must exist an inverse system with impulse response $h_2[n]$ such that $h_1[n] * h_2[n] = \delta[n]$.

Note:

i. If $x_1[n]$ and $x_2[n]$ are both causal, then $x_1[n]*x_2[n]$ is also causal.

ii. If $x_1[n]$ is of length N_1 and $x_2[n]$ is of length N_2, then their convolution $y[n] = x_1[n]*x_2[n]$ has length N_1+N_2-1.

Example 1.25:

Find the output of LTI system having impulse response $h[n] = \left(\frac{1}{2}\right)^n u[n]$ and input $x[n] = 5e^{i\pi n/3}$ for $-\infty < n < \infty$.

[2080 Baishakh]

Solution:

Given,

$x[n] = 5e^{i\pi n/3}$; for $-\infty < n < \infty$

$h[n] = \left(\frac{1}{2}\right)^n u[n]$; for $-\infty < n < \infty$

We know

$y[n] = x[n] * h[n]$ or $h[n] * x[n]$ [Commutative property]

$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u[k]\, 5e^{i(n-k)\pi/3}$$

$$u[k] = \begin{cases} 1; & \text{for } k \geq 0 \\ 0; & \text{for } k < 0 \end{cases}$$

$$y[n] = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k 5\, e^{i\pi n/3}\, e^{-i\pi k/3}$$

$$= 5e^{i\pi n/3} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k (e^{-i\pi/3})^k$$

$$= 5e^{i\pi n/3} \sum_{k=0}^{\infty} \left(\frac{1}{2} e^{-i\pi/3}\right)^k$$

$$= 5e^{i\pi n/3} \frac{1}{1-\frac{1}{2}e^{-i\pi/3}} \quad \left[\because \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}\right]$$

$$= 5e^{i\pi n/3}(1 - i0.577)$$

$$\therefore\ y[n] = 5(1 - i0.577)\, e^{i\pi n/3}$$

which is the required output of LTI system.

Example 1.26:

Find the output of LTI system having input signal $x[n] = \delta[n+2] + \delta[n-1] - \delta[n-3]$ and $h[n] = 2\delta[n+1] + 2\delta[n-1]$.

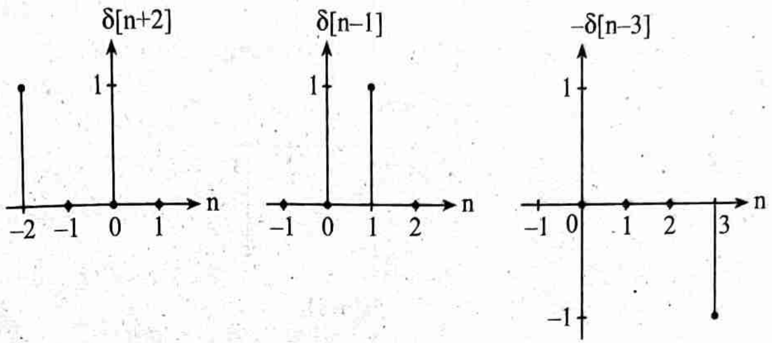
Solution: **[2079 Bhadra]**

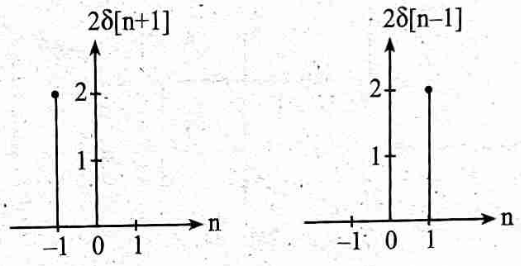
Given,

$x[n] = \delta[n+2] + \delta[n-1] - \delta[n-3]$

$h[n] = 2\delta[n+1] + 2\delta[n-1]$

In graphical representation,





So, $x[n] = \{1, 0, \underset{\uparrow}{0}, 1, 0, -1\}$

$h[n] = \{2, \underset{\uparrow}{0}, 2\}$

Now, $y[n] = h[n] * x[n]$

$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

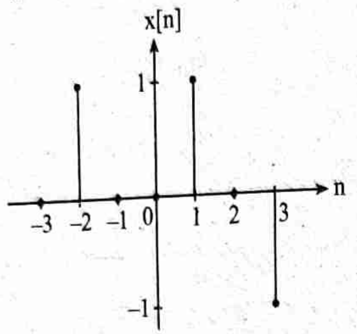
$$= \sum_{k=-1}^{1} h[k]\, x[n-k]$$

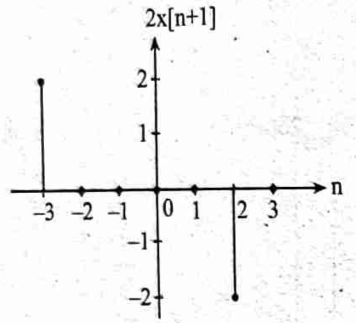
$$= h[-1]\, x[n+1] + h[0]\, x[n] + h[1]\, x[n-1]$$

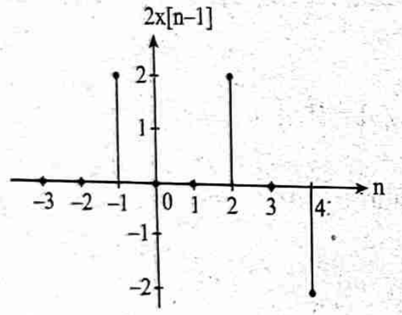
$$= 2x[n+1] + 0 \times x[n] + 2x[n-1]$$

$$\therefore\ y[n] = 2x[n+1] + 2x[n-1]$$

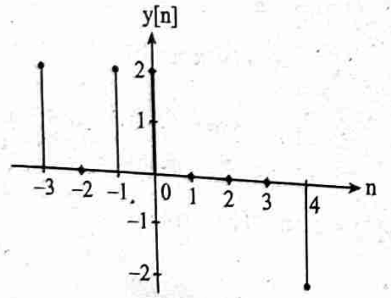
In graphical representation,







Therefore,



Hence, $y[n] = \{2, 0, 2, \underset{\uparrow}{2}, 0, 0, 0, -2\}$

Example 1.27:

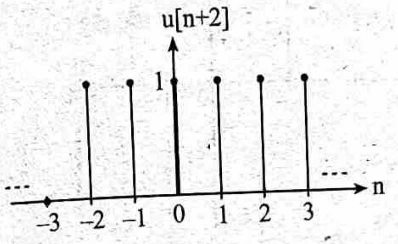
Find the output of LTI system having impulse response h[n] $= \left(\frac{1}{2}\right)^n$ {u[n+2] – u[n–2]} to the input x[n] = {2,1,0,–1,4}.

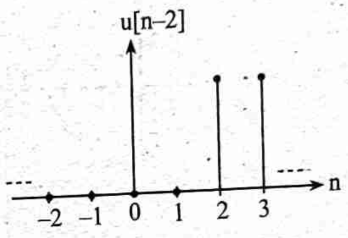
[2079 Baishakh]

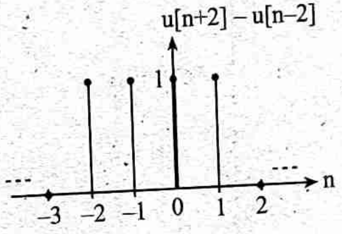
Solution:

Given, $x[n]=\{2, 1, 0, -1, 4\}$ and $h[n]=\left(\frac{1}{2}\right)^n \{u[n+2]-u[n-2]\}$

In graphical representation,







So, $h[n] = \left(\frac{1}{2}\right)^n \quad ; -2 \le n \le 1$

$= \{4, 2, \underset{\uparrow}{1}, 0.5\}$

Now, $y[n] = h[n] * x[n]$

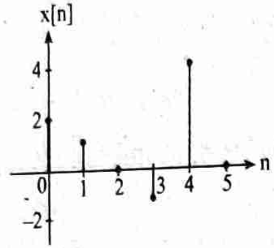
$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

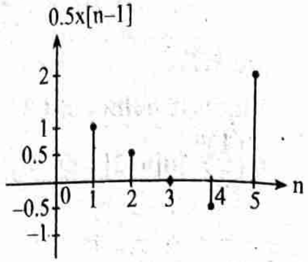
$$= \sum_{k=-2}^{1} h[k]\, x[n-k]$$

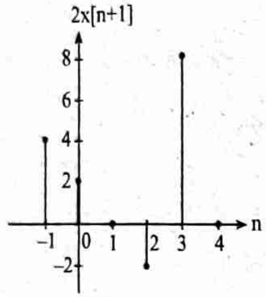
$=h[-2]x[n+2]+h[-1]x[n+1]+h[0]x[n]+h[1]x[n-1]$

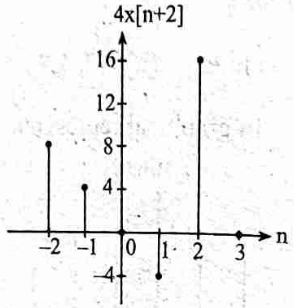
or, $y[n] = 4x[n+2] + 2x[n+1] + x[n] + 0.5x[n-1]$

In graphical representation,

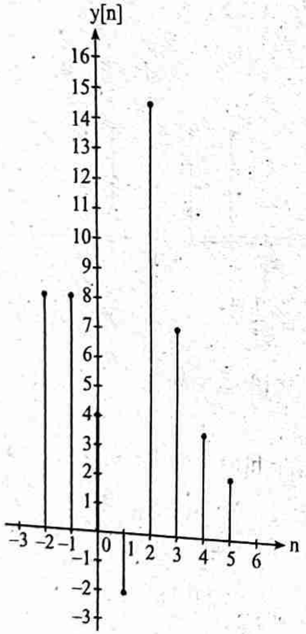








Therefore,



$\therefore\ y[n] = \{8, 8, \underset{\uparrow}{4}, -2, 14.5, 7, 3.5, 2\}$

Example 1.28:

Find the output of LTI system having impulse response $h[n] = u[n] - u[n-4]$ and input signal $x[n] = \left(\frac{1}{2}\right)^n u[n]$.

[2078 Bhadra]

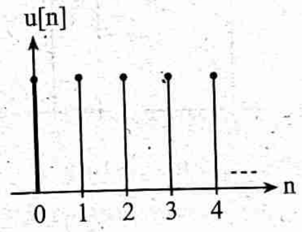
Solution:

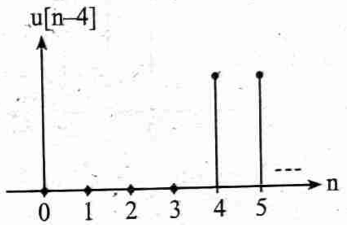
Given,

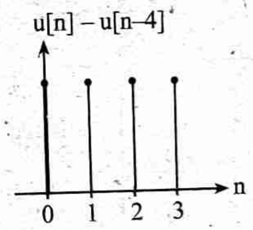
$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$h[n] = u[n] - u[n-4]$$

In graphical representation,







So, $h[n] = \{\underset{\uparrow}{1}, 1, 1, 1\};\ 0 \le n \le 3$

Now,

$$y[n] = h[n] * x[n]$$

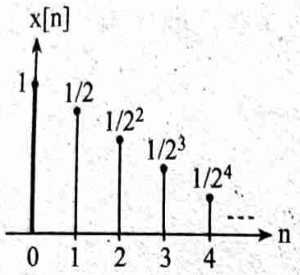
$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

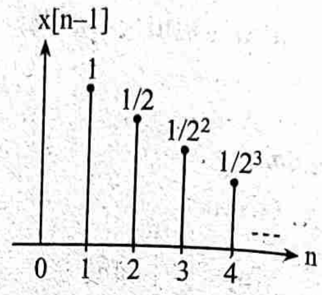
$$= \sum_{k=0}^{3} h[k]\, x[n-k]$$

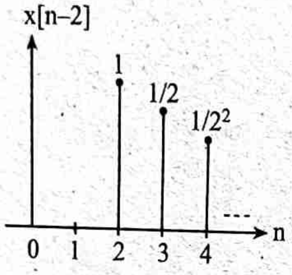
$$= h[0]\, x[n] + h[1]\, x[n-1] + h[2]\, x[n-2] + h[3]\, x[n-3]$$

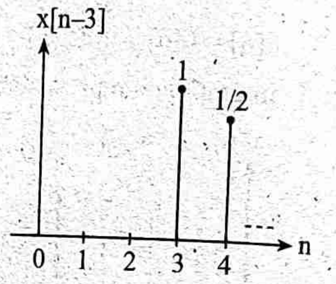
$$\therefore\ y[n] = x[n] + x[n-1] + x[n-2] + x[n-3]$$

In graphical representation,

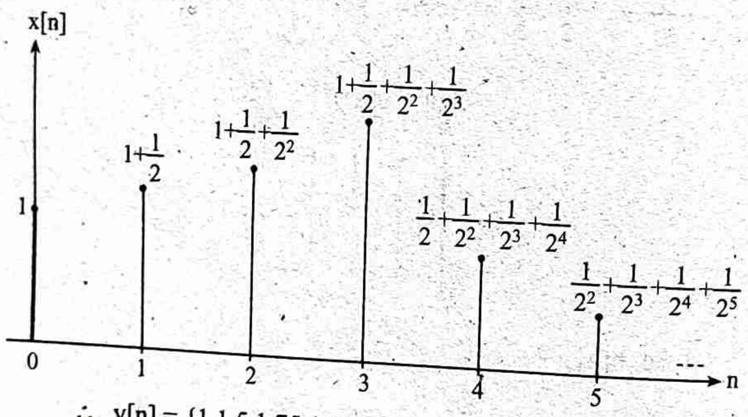








Therefore,



$$\therefore\ y[n] = \{1, 1.5, 1.75, 1.875, 0.9375, 0.46875, 0.234375, ...\}$$

Example 1.29:

Find the output of LTI system having impulse response $h[n]$ with $h[-2] = 3$, $h[0] = 2$, $h[1] = 1$ and input signal $x[n] = (2)^n$, for $-1 \le n \le 3$. Also, check the answer.

[2075 Chaitra]

Solution:

Given,

$$x[n] = (2)^n \text{ for } -1 \le n \le 3$$

$$\therefore\ x[n] = \{0.5, 1, 2, 4, 8\}$$

$$\text{and, } h[n] = \{3, 0, \underset{\uparrow}{2}, 1\}$$

Now,

$$y[n] = h[n] * x[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$= \sum_{k=-2}^{1} h[k]\, x[n-k]$$

$$= h[-2]x[n+2] + h[-1]x[n+1] + h[0]x[n] + h[1]x[n-1]$$

$$= 3x[n+2] + 0\times x[n+1] + 2x[n] + 1\times x[n-1]$$

$$\therefore\ y[n] = 3x[n+2] + 2x[n] + x[n-1]$$

Solve like previous one.

1.9 Frequency Response of LTI System

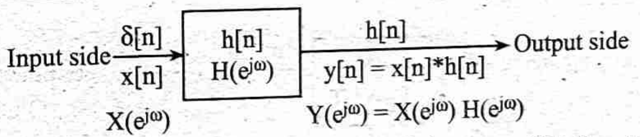


Fig.: Relationship between inputs and outputs for on LTI discrete-time system

It input x(n) is applied to a discrete-time system h[n], we get output y[n]. By applying convolution property, we have

$$y[n] = x[n] * h[n]$$

Hence,

$$Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$$

$$\therefore\ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$H(e^{j\omega})$ is called *discrete-time system's frequency response*. $H(e^{j\omega})$ is, in general, a complex function. Hence, we can write

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\theta(\omega)}$$

Then,

$|H(e^{j\omega})|$ is called *magnitude response* of the system.

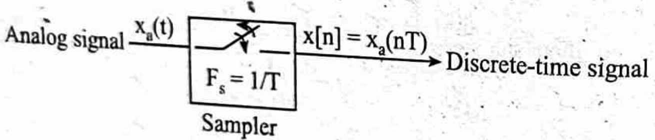
$\theta(\omega)$ is called *phase response* of the system.

1.10 Sampling of Continuous Time Signal, Spectral Properties of Sampled Signal

Sampling of Continuous Time (Analog) Signal

Most of the signals that we use in practical life are continuous-time signals such as speech signals, the sesmic signals and the image signals. When these signals are to be transmitted over a distance, or stored, or are to be processed in some manner, a lot of advantages are possible if these signals are first converted into discrete/digital signals for various operations.

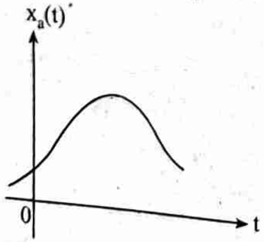
Sampling is a process of converting continuous time signal (CTS) to discrete time signal (DTS) by taking samples at discrete instances of time.



F_s = Sampling frequency (sampling rate)

T = Sampling interval

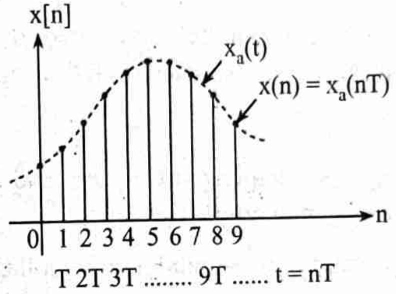
Let us consider an analog signal $x_a(t)$.



Suppose x[n] be the discrete time signal obtained by taking samples of the analog signal $x_a(t)$ every T seconds.

We have,

$$x(n) = x_a(nT) ; -\infty < n < \infty$$



The time interval T between successive samples is called *sampling period* or *sampling interval* and its reciprocal $\frac{1}{T} = F_s$ is called the *sampling rate*.

Let $x_a(t) = A\cos(2\pi Ft + \theta)$ .... (i) be an analog signal.

If sampled periodically at the rate $\frac{1}{T}$ samples per second, yields.

$$x_a(nT) = x(n) = A\cos(2\pi FnT + \theta) \text{ ....(ii)}$$

$$\text{or, } x(n) = A\cos\left(\frac{2\pi nF}{F_s} + \theta\right) \text{ .... (iii)}$$

We know that,

$$x(n) = A\cos(\omega n + \theta) = A\cos(2\pi fn + \theta) \text{ .... (iv)}$$

Comparing equation (iv) and equation (iii), we get

$$f = \frac{F}{F_s}$$

f = Normalized frequency of DT signal

F = Frequency of continuous time (CT) signal

F_s = Sampling frequency

Note: To convert CT signal to DT signal

i. Divide by F_s

ii. Change t to n

Sampling Theorem

Any CT signal can be represented by its samples and original signal can be reconstructed from the samples if

$$F_s \geq 2F_m$$

where F_s is sampling frequency and F_m is maximum frequency component in original signal.

Nyquist rate: It is the minimum sampling frequency and is defined as

$$F_s = 2F_m$$

Nyquist interval: It is the maximum time interval and is defined as $T_s = \frac{1}{F_s} = \frac{1}{2F_m}$.

Example 1.30:

Find the corresponding discrete time signal for the given continuous time signal $x_1(t) = \cos 2\pi(10)t$ sampled at F_s = 40 Hz.

Solution:

We know that,

$$f = \frac{F}{F_s} = \frac{10}{40} = \frac{1}{4}$$

Replacing the value in the equation,

$$x_1(n) = \cos(2\pi f n)$$

$$= \cos\left(\frac{\pi}{2} n\right)$$

Example 1.31:

Find the corresponding discrete time signal for the given continuous time signal $x_2(t) = \cos 2\pi(50)t$ sampled at F_s = 40 Hz.

Solution:

We know,

$$f = \frac{F}{F_s} = \frac{50}{40} = \frac{5}{4}$$

Replacing the value in the equation,

$$x_2(n) = \cos(2\pi f n)$$

$$= \cos\left(\frac{5\pi}{2} n\right)$$

$$= \cos\left[\left(2\pi + \frac{\pi}{2}\right) n\right]$$

$$= \cos\left(\frac{\pi}{2} n\right)$$

From last two examples, we can observe that the discrete time signal (sampled signal) for CT signal $x_1(t) = \cos 2\pi(10)t$ and $x_2(t) = \cos 2\pi(50)t$ is identical i.e. $x_1(n) = x_2(n) = \cos\left(\frac{\pi}{2} n\right)$. If we are given the sampled signal, we will not be able to distinguish if the sample signal is due to $x_1(t)$ or $x_2(t)$. We say that the frequency F_2 = 50 Hz is the alias of the frequency F_1 = 10 Hz at the sampling frequency of F_s = 40Hz.

Example 1.32:

Consider the analog signal: $x_a(t) = 3\cos 100\pi t$

a. **Determine the minimum sampling frequency to avoid aliasing.**
b. **Suppose that the signal is sampled at the rate F_s = 200Hz. What is the discrete time signal obtained after sampling?**
c. **Suppose that the signal is sampled at the rate F_s = 75 Hz. What is the discrete time signal obtained after sampling?**
d. **What is the frequency $0 < F < F_s/2$ of a sinusoid that yield samples identical to those obtained in (c).**

Solution:

We have, F = 50 Hz [From question]

a. $F_s = 2F_m$

$= 2 \times 50$

$= 100$ Hz [Minimum sampling frequency]

b. Here, $F_s = 200$ Hz

$$x[n] = 3\cos\left(2\pi \frac{F}{F_s} n\right)$$

$$= 3\cos\left(2\pi \frac{50}{200} n\right)$$

$$= 3\cos\left(\frac{\pi}{2} n\right)$$

c. Here, $F_s = 75$ Hz

$$x[n] = 3\cos\left(2\pi \frac{50}{75} n\right)$$

$$= 3\cos\left(\frac{4\pi}{3} n\right)$$

$$= 3\cos\left[\left(2\pi - \frac{2\pi}{3}\right) n\right]$$

$$= 3\cos\left(\frac{2\pi}{3} n\right)$$

d. $f = \frac{1}{3}$ (from c)

and, $F_s = 75$ Hz

$\therefore \; F = f F_s = \frac{1}{3} \times 75 = 25$ Hz

So, $x(t) = 3\cos 2\pi(25)t$ $\quad$ $[\because A\cos(2\pi f t)]$

$= 3\cos(50\pi t)$

Example 1.33:

Consider an analog signal, $x_a(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t$.

a. What is the Nyquist rate for this signal?

b. Obtain the DT signal after sampling at Nyquist rate.

Solution:

From the given signal $x_a(t)$, we have

$F_1 = 25$ Hz, $F_2 = 150$ Hz, and $F_3 = 50$ Hz

Thus, we can say $F_{max} = 150$ Hz

a. According to sampling theorem,

$F_s = 2F_{max} = 2 \times 150$ Hz $= 300$ Hz

Hence, the Nyquist rate is 300 Hz for this signal.

b. The DT signal after sampling at Nyquist rate is

$$x[n] = 3\cos\frac{50\pi}{300} n + 10\sin\frac{300\pi}{300} n - \cos\frac{100\pi}{300} n$$

$$= 3\cos\left(\frac{\pi}{6} n\right) + 10\sin(\pi n) - \cos\left(\frac{\pi}{3} n\right)$$

Observation: The sampled signal is x[n] for the original signal $x_a(t)$. Here, we can observe that the signal component $10\sin 300\pi t$ sampled at the Nyquist rate 300 Hz results in the sample $10\sin(\pi n)$, which is identically zero. This is because we are sampling the analog sinusoid at its zero crossing point and we miss the signal component completely.

The remedy is to sample the signal at the rate higher than the Nyquist rate.

Example 1.34:

Consider the signal, $x_a(t) = 3\cos 2000\pi t + 5\sin 6000\pi t + 10\cos 12000\pi t$.

a. What is the Nyquist rate for this signal?

b. If we sample the signal at 5 KHz, what is the DT signal obtained after sampling?

c. What is the analog signal obtained after we reconstruct the DT signal obtained above using ideal interpolation?

Solution:

From the given signal, we have

$F_1 = 1000$ Hz, $F_2 = 3000$ Hz, $F_3 = 6000$ Hz

Thus, we can say $F_{max} = 6000$ Hz

a. According to sampling theorem,

Nyquist rate = $2F_{max}$ = 12000 Hz = 12 KHz

b. When sampled at 5 KHz, i.e. F_s = 5 KHz, the DT signal is

$$x[n] = 3\cos\left(\frac{2000}{5000}\pi n\right)+5\sin\left(\frac{6000}{5000}\pi n\right)+10\cos\left(\frac{12000}{5000}\pi n\right)$$

$$\text{or, } x[n] = 3\cos\left(\frac{2\pi n}{5}\right) + 5\sin\left(\frac{6\pi n}{5}\right) + 10\cos\left(\frac{12\pi n}{5}\right)$$

$$=3\cos\left(\frac{2\pi n}{5}\right)+5\sin\left[\left(2\pi-\frac{4\pi}{5}\right)n\right]+10\cos\left[\left(2\pi+\frac{2\pi}{5}\right)n\right]$$

$$= 3\cos\left(\frac{2\pi}{5}n\right) + 5\sin\left(\frac{4\pi}{5}n\right) + 10\cos\left(\frac{2\pi}{5}n\right)$$

$$= 13\cos\left(\frac{2\pi}{5}n\right) + 5\sin\left(\frac{4\pi}{5}n\right)$$

$$\therefore\ x[n] = 13\cos\left(2\pi\times\frac{1}{5}n\right) + 5\sin\left(2\pi\times\frac{2}{5}n\right)$$

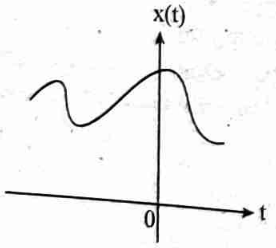
c. $x(t) = 13\cos\left(2\pi\times\frac{1}{5}\times5000t\right) + 5\sin\left(2\pi\times\frac{2}{5}\times5000t\right)$

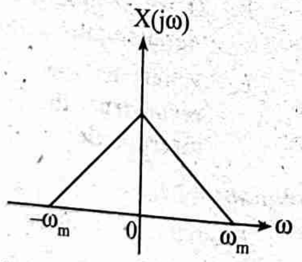
$$= 13\cos(2000\pi t) + 5\sin(4000\pi t)$$

Here, the signal is undersampled, so the original signal cannot be reconstructed.

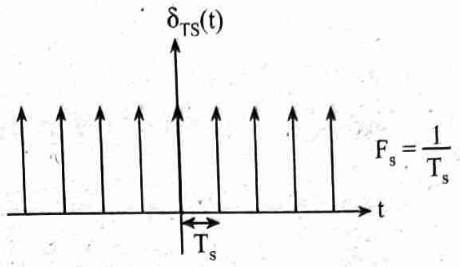
Sampling CT signal and Spectral Property of Sampled Signal

Let us consider a band limited signal x(t) which is limited to ω_m.



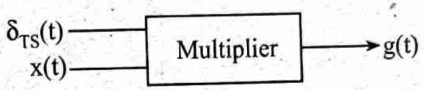


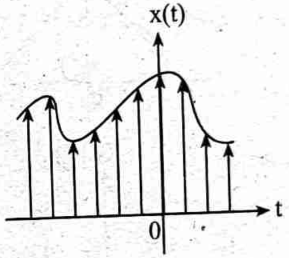
Now, to find the samples of x(t), we multiply x(t) with impulse train $\delta_{TS}(t)$.



Mathematically,

$$\delta_{TS}(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT_s).$$





Fourier series in trigonometric form is

$$\delta_{TS}(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_s t)$$

where, $a_0 = \frac{1}{T_s}$ and $a_n = \frac{2}{T_s}$

$$\text{or, } \delta_{TS}(t) = \frac{1}{T_s}+\frac{2}{T_s}\cos(\omega_s t)+\frac{2}{T_s}\cos(2\omega_s t)+\frac{2}{T_s}\cos(3\omega_s t) + \ldots\ldots$$

$$\text{or, } \delta_{TS}(t) = \frac{1}{T_s}[1+2\cos(\omega_s t)+2\cos(2\omega_s t)+2\cos(3\omega_s t) + \ldots\ldots\ldots]$$

$$\text{or, } g(t) = \delta_{TS}(t)x(t)$$

$$= \frac{1}{T_s}[x(t) + 2x(t)\cos(\omega_s t) + 2x(t)\cos(2\omega_s t) + 2x(t)\cos(3\omega_s t) + \ldots\ldots\ldots]$$

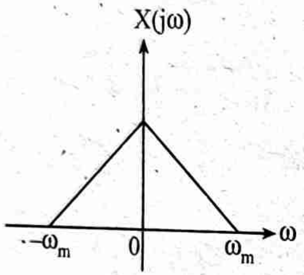
As $x(t) \rightarrow X(j\omega)$,

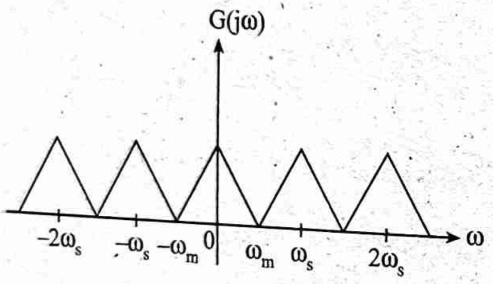
$$2x(t)\cos(\omega_s t) \rightarrow X(j\omega - j\omega_s) + X(j\omega + j\omega_s)$$

$$2x(t)\cos(2\omega_s t) \rightarrow X(j\omega - 2j\omega_s) + X(j\omega + 2j\omega_s)$$

$$\therefore \; G(j\omega) = \frac{1}{T_s}[X(j\omega) + X(j\omega - j\omega_s) + X(j\omega + j\omega_s) + X(j\omega - 2j\omega_s) + X(j\omega + 2j\omega_s) + \ldots\ldots]$$

$$\therefore \; G(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j\omega - jk\omega_s)$$





To reconstruct x(t) from g(t), we need to recover X(jω) from G(jω). This recovery is only possible if there is no overlapping between successive cycles of G(jω).

For no overlapping,

$\omega_s \geq 2\omega_m$

i.e., $f_s \geq 2f_m$

Once the condition is met, X(jω) can be recovered from G(jω) using low pass filter.

Chapter – 2

Z-TRANSFORM

2.1 Definition, Convergence of Z-transform and Region of Convergence

Definition

Z-transform of the DT signal x[n] is defined as the power series,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

where z is a complex variable. It can be expressed in polar form as

$$z = re^{j\omega}$$

where r is the magnitude of z and ω is phase angle in radian.

Z-transform makes the analysis, designing and realization of DT signal and LTI system very easy. It converts difference equations into algebraic equations, which are easy to manipulate and solve.

We can obtain x(n) from X(z) and the process is called *inverse z-transform*.

$$x(n) \xleftrightarrow{ZT} X(z)$$

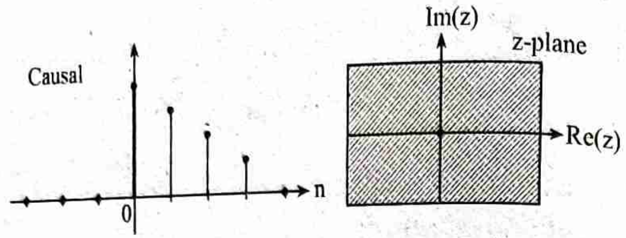
Region of Convergence (ROC)

Region of convergence (ROC) of X(z) is the value of z for which the X(z) converges/results in a finite value.

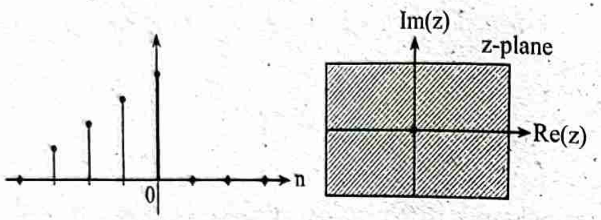
Properties of region of convergence (ROC):

i. The region of convergence (ROC) does not contain any pole of X(z).

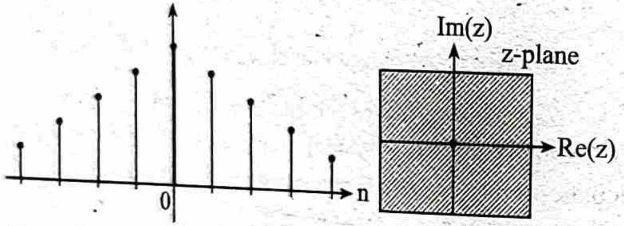
ii. For a finite duration causal signal, the ROC is entire z-plane except z = 0.



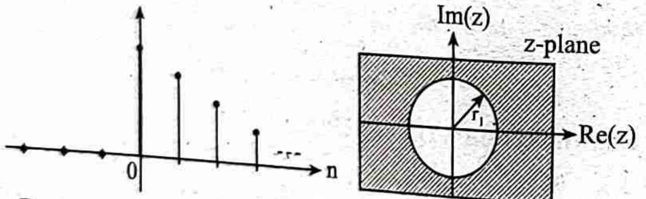
iii. For a finite duration anti-causal signal, the ROC is entire z-plane except $z = \infty$.



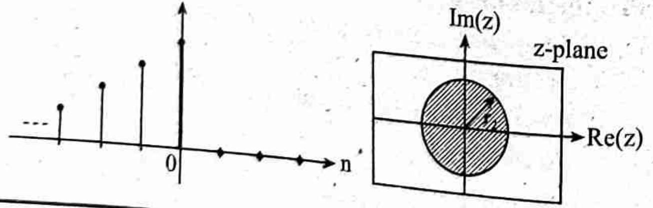
iv. For a finite duration two-sided signal, the ROC is entire z-plane except $z = 0$ and $z = \infty$.



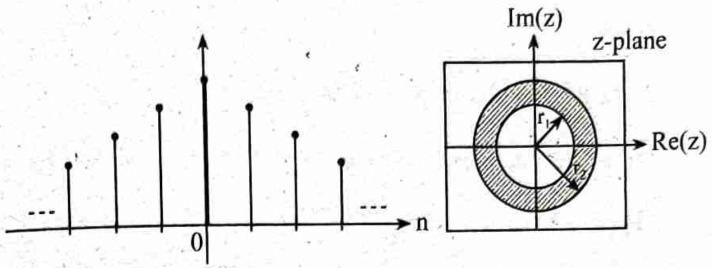
v. For an infinite duration causal signal, the ROC is $|z| > r_1$.



vi. For an infinite duration anti-causal signal, the ROC is $|z|<r_2$.



vii. For an infinite duration two sided signal, the ROC is $r_1<|z|<r_2$.



2.2 Properties of Z-Transform

1. Linearity:

If $x_1(n) \xleftrightarrow{ZT} X_1(z)$; ROC: R_1

and, $x_2(n) \xleftrightarrow{ZT} X_2(z)$; ROC: R_2

Then,

$$ax_1(n) + bx_2(n) \xleftrightarrow{ZT} aX_1(z) + bX_2(z) \quad ; \text{ROC: } R_1 \cap R_2$$

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [ax_1(n) + bx_2(n)]z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} ax_1(n)z^{-n} + \sum_{n=-\infty}^{\infty} bx_2(n)z^{-n}$$

$$= a \sum_{n=-\infty}^{\infty} x_1(n)z^{-n} + b \sum_{n=-\infty}^{\infty} x_2(n)z^{-n}$$

$$= aX_1(z) + bX_2(z)$$

ROC of overall transform is the intersection of the ROC of the individual transform.

2. Time shifting:

If $x(n) \xleftrightarrow{ZT} X(z)$; ROC : R

Then,

$$x(n-k) \xleftrightarrow{ZT} z^{-k}X(z) \qquad ; \text{ROC: } R' = R$$

Proof:

$$ZT\,[x(n-k)] = \sum_{n=-\infty}^{\infty} x(n-k)z^{-n}$$

Put $n - k = n'$, then $n = n' + k$

$$ZT[x(n-k)] = \sum_{n'=-\infty}^{\infty} x(n')\, z^{-(n'+k)}$$

$$= \sum_{n'=-\infty}^{\infty} x(n')\, z^{-n'} z^{-k}$$

$$= z^{-k} \sum_{n'=-\infty}^{\infty} x(n')z^{-n'}$$

$$= z^{-k}X(z)$$

ROC of the overall transform is same as that of x[n] except for $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$.

3. Scaling in the z-domain:

If $x(n) \xleftrightarrow{ZT} X(z) \qquad ;\text{ROC} : R = r_1 < |z| < r_2$

Then, $a^n x(n) \xleftrightarrow{ZT} X(a^{-1}z);\text{ROC: } R' = |a|r_1 < |z| < |a|r_2$

Proof:

$$ZT[x(n)] = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = X(z)$$

$$\text{or, } ZT[a^n x(n)] = \sum_{n=-\infty}^{\infty} a^n x(n)z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\,(az^{-1})^{n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\,(a^{-1}z)^{-n}$$

$$= X(a^{-1}z)$$

Also, the ROC of X(z) is $r_1 < |z| < r_2$

So, the ROC of $X(a^{-1}z)$ is $r_1 < |a^{-1}z| < r_2$

$$\Rightarrow |a|r_1 < |z| < |a|r_2$$

4. Time reversal:

If $x(n) \xleftrightarrow{ZT} X(z) \qquad ; \text{ROC: } r_1 < |z| < r_2$

Then,

$$x(-n) \xleftrightarrow{ZT} X(z^{-1}) \qquad ; \text{ROC: } \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

Proof:

$$ZT[x(n)] = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = X(z)$$

$$ZT[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n)z^{-n}$$

Let $l = -n$, then,

$$ZT[x(-n)] = \sum_{l=\infty}^{-\infty} x(l)\, z^{l}$$

$$= \sum_{l=-\infty}^{\infty} x(l)\, (z^{-1})^{-l}$$

$$= X(z^{-1})$$

Also, the ROC of X(z) is $r_1 < |z| < r_2$

So, the ROC of $X(z^{-1})$ is $r_1 < |z^{-1}| < r_2$

$$\Rightarrow \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

5. Differentiation in z-domain:

If $x(n) \xleftrightarrow{ZT} X(z) \qquad ; \text{ROC: } r_1 < |z| < r_2$

Then,

$$nx(n) \xleftrightarrow{ZT} -z\frac{dX(z)}{dz} \qquad ; \text{ROC: } r_1 < |z| < r_2$$

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$\text{or, } \frac{dX(z)}{dz} = \frac{d}{dz}\left(\sum_{n=-\infty}^{\infty} x(n)z^{-n}\right)$$

$$= \sum_{n=-\infty}^{\infty} -nx(n)z^{-n-1}$$

$$= \sum_{n=-\infty}^{\infty} -nx(n)z^{-n}z^{-1}$$

$$= \frac{-1}{z} \sum_{n=-\infty}^{\infty} nx(n)z^{-n}$$

$$\text{or, } \frac{d}{dz}X(z) = -z^{-1}\, ZT[nx(n)]$$

$$\therefore\ ZT[nx(n)] = -z\frac{dX(z)}{dz}$$

6. **Convolution:**

$$\text{If } x_1(n) \xleftrightarrow{ZT} X_1(z)$$

$$\text{and, } x_2(n) \xleftrightarrow{ZT} X_2(z)$$

Then,

$$x_1(n) * x_2(n) \xleftrightarrow{ZT} X_1(z)X_2(z) = X(z)$$

ROC of X(z) is intersection of ROC of the individual transform $X_1(z)$ and $X_2(z)$.

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$\text{and, } x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

Now,

$$ZT[x_1(n) * x_2(n)] = X(z)$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \sum_{n=-\infty}^{\infty} x_2(n-k)z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \sum_{n=-\infty}^{\infty} x_2(n-k)z^{-(n-k)}\, z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)z^{-k} \sum_{n=-\infty}^{\infty} x_2(n-k)z^{-(n-k)}$$

$$= X_1(z)\, X_2(z)$$

Poles and Zeros of Z-Transform

Z-transform can be represented as the ratio of two polynomials in z^{-1} (or z).

$$X(z) = \frac{N(z)}{D(z)}$$

$$= \frac{b_0 + b_1z^{-1} + b_2z^{-2} + \ldots\ldots + b_mz^{-m}}{a_0 + a_1z^{-1} + a_2z^{-2} + \ldots\ldots + a_mz^{-m}}$$

$$= \frac{\sum_{k=0}^{m} b_k z^{-k}}{\sum_{k=0}^{m} a_k z^{-k}}$$

Poles: Values of z for which X(z) = ∞ are called *poles* of X(z).

Zeros: Values of z for which X(z) = 0 are called *zeros* of X(z).

Some Standard Z-Transforms

S.N.	Function, x(n)	X(z)	ROC
i.	$\delta(n)$	1	All z
ii.	$u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
iii.	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
iv.	$a^{-n}u(n)$	$\frac{1}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > \left\lvert\frac{1}{a}\right\rvert$
v.	$b^n u(-n-1)$	$\frac{-1}{1-bz^{-1}}$	$\lvert z\rvert < \lvert b\rvert$
vi.	$na^n u(n)$	$\frac{az}{(z-a)^2}$	$\lvert z\rvert > a$
vii.	$na^n u(-n-1)$	$\frac{-az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < a$

Example 2.1:

Locate the ROC of the following signal.

$x[n] = (0.6)^n u[n] + (0.25)^n u[n]$. **[2075 Ashwin]**

Solution:

Given signal is

$x[n] = (0.6)^n u[n] + (0.25)^n u[n]$

Z-transform of given signal is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [(0.6)^n u[n] + (0.25)^n u[n]]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} (0.6)^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} (0.25)^n u[n] z^{-n}$$

$$u[n] = \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$

$$\text{So, } X(z) = \sum_{n=0}^{\infty} (0.6)^n z^{-n} + \sum_{n=0}^{\infty} (0.25)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (0.6z^{-1})^n + \sum_{n=0}^{\infty} (0.25z^{-1})^n$$

$$\text{Also, } \sum_{n=0}^{\infty} a^n = \frac{1}{1-a};\ |a| < 1$$

$$\text{or, } X(z) = \frac{1}{1-0.6z^{-1}} + \frac{1}{1-0.25z^{-1}} = X_1(z) + X_2(z)$$

For $X_1(z)$, ROC: $|0.6z^{-1}| < 1$

$\Rightarrow |z| > 0.6$

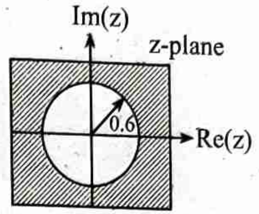
For $X_2(z)$, ROC: $|0.25z^{-1}| < 1$

$\Rightarrow |z| > 0.25$

Therefore,

$$X(z) = \frac{1}{1-0.6z^{-1}} + \frac{1}{1-0.25z^{-1}}$$

and, region of convergence: $|z| > 0.6$



Example 2.2:

Find z-transform and locate the ROC of the following signal.

$x[n] = a^n u[n] + b^n u[-n-1]$ **[2075 Chaitra, 2070 Chaitra]**

Solution:

Given signal is,

$x[n] = a^n u[n] + b^n u[-n-1]$

The z-transform of given signal is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [a^n u[n] + b^n u[-n-1]]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} b^n u[-n-1] z^{-n}$$

$$= X_1(z) + X_2(z)$$

For $X_1(z)$,

$$X_1(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n}$$

$$\text{We have, } u[n] = \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$

$$\text{So, } X_1(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (a^n z^{-1})^n$$

We know,

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad ; |a| < 1$$

$$\text{So, } X_1(z) = \sum_{n=0}^{\infty} (a^n z^{-1})^n = \frac{1}{1-az^{-1}}$$

Also, $|az^{-1}| < 1$

$\Rightarrow |z| > |a|$

$\therefore$ ROC: $|z| > |a|$

For $X_2(z)$,

$$X_2(z) = \sum_{n=-\infty}^{\infty} b^n u[-n-1] z^{-n}$$

$$\text{We have, } u[-n-1] = \begin{cases} 1, & \text{for } n < 0 \\ 0, & \text{for } n \geq 0 \end{cases}$$

$$\text{So, } X_2(z) = \sum_{n=-\infty}^{-1} b^n z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (bz^{-1})^n$$

Put n = –n,

$$X_2(z) = \sum_{n=\infty}^{1} (b^{-1}z)^n$$

$$= \sum_{n=1}^{\infty} (b^{-1}z)^n$$

$$= \sum_{n=0}^{\infty} (b^{-1}z)^n - 1$$

$$= \frac{1}{1-b^{-1}z} - 1$$

$$= \frac{b^{-1}z}{1-b^{-1}z}$$

$$= \frac{1}{bz^{-1}-1}$$

$$= \frac{-1}{1-bz^{-1}}$$

Also, $|b^{-1}z| < 1 \Rightarrow |z| < |b|$

$\therefore$ ROC: $|z| < |b|$

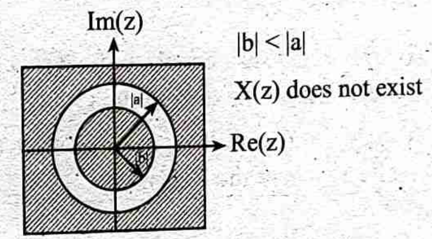
Therefore,

$$X(z) = X_1(z) + X_2(z)$$

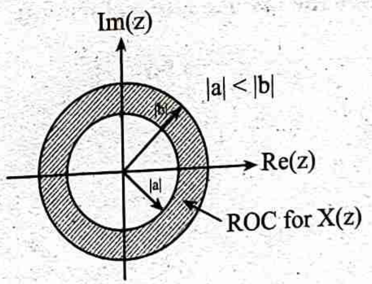
$$= \frac{1}{1-az^{-1}} - \frac{1}{1-bz^{-1}}$$

And, ROC is given by the intersection of individual ROC of two signals.

- If a = b, there will be no intersection so there will be no region of convergence.
- If a > b, there is no common ROC so the z-transform does not converge and X(z) does not exist.



- If a < b, there is a common region of convergence (ROC) and the z-transform converges, so X(z) exists.



$$\text{Hence, } X(z) = \frac{1}{1-az^{-1}} - \frac{1}{1-bz^{-1}} \text{ if ROC: } |a| < |z| < |b|$$

2.3 Inverse Z-transform by Long Division and Partial Fraction Expansion

We can recover the original signal x[n] using the inverse z-transform of X(z). The most common methods used are:

i. By long division

ii. By partial fraction

i. By Long Division

Given, $X(z) = \frac{N(z)}{D(z)}$, we divide N(z) by D(z) and get the quotient as a series in powers of (z^{-1}). We try to obtain a form

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

where x(n) is the coefficient sequence of various powers of (z^{-1}) in the quotient.

Note:

- To solve causal signal, we start the divisor with highest term at the beginning.
- To solve anti-causal signal, we start the divisor with smallest term at the beginning.

Example 2.3:

Determine the inverse z-transform of:

$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$ **for**

i. $|z| > 1$

ii. $|z| < 0.5$

Solution:

Given,

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

i. Here,

for $|z| > 1$ (casual signal)

$$\begin{array}{l}
1-1.5z^{-1}+0.5z^{-2}\overline{)\,1\qquad}\left(1+\frac{3}{2}z^{-1}+\frac{7}{4}z^{-2}+\frac{15}{8}z^{-3}+\frac{31}{16}z^{-4}+\frac{63}{32}z^{-5}\right.\\
\left(1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}\right)\ \underline{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}\\
\qquad\qquad\qquad \frac{3}{2}z^{-1}-\frac{1}{2}z^{-2}\\
\qquad\qquad\qquad \underline{\frac{3}{2}z^{-1}-\frac{9}{4}z^{-2}+\frac{3}{4}z^{-3}}\\
\qquad\qquad\qquad\qquad \frac{7}{4}z^{-2}-\frac{3}{4}z^{-3}\\
\qquad\qquad\qquad\qquad \underline{\frac{7}{4}z^{-2}-\frac{21}{8}z^{-3}+\frac{7}{8}z^{-4}}\\
\qquad\qquad\qquad\qquad\qquad \frac{15}{8}z^{-3}-\frac{7}{8}z^{-4}\\
\qquad\qquad\qquad\qquad\qquad \underline{\frac{15}{8}z^{-3}-\frac{45}{16}z^{-4}+\frac{15}{16}z^{-5}}\\
\qquad\qquad\qquad\qquad\qquad\qquad \frac{31}{16}z^{-4}-\frac{15}{16}z^{-5}\\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{\frac{31}{16}z^{-4}-\frac{93}{32}z^{-5}+\frac{31}{32}z^{-6}}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \frac{63}{32}z^{-5}-\frac{31}{32}z^{-6}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{\frac{63}{32}z^{-5}-\frac{189}{64}z^{-6}+\frac{63}{64}z^{-7}}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \frac{127}{64}z^{-6}-\frac{63}{64}z^{-7}
\end{array}$$

$$\therefore\ X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

$$= 1 + \frac{3}{2}z^{-1} + \frac{7}{4}z^{-2} + \frac{15}{8}z^{-3} + \frac{31}{16}z^{-4} + \frac{63}{32}z^{-5} + \text{.......................}$$

Comparing with power series for causal signal,

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

$$= x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + \text{........}$$

$$\therefore\ x[n] = \left\{\underset{\uparrow}{1}, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \frac{63}{32}, \text{......}\right\}$$

ii. Here, for $|z| < 0.5$ (anti-casual signal),

$$
\begin{array}{l}
\frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1\,\big)\,1 \quad \big(2z^2+6z^3+14z^4+30z^5+62z^6+126z^7+....\\
\qquad\qquad \underline{1-3z+2z^2}\\
\qquad\qquad\quad 3z-2z^2\\
\qquad\qquad\quad \underline{3z-9z^2+6z^3}\\
\qquad\qquad\qquad 7z^2-6z^3\\
\qquad\qquad\qquad \underline{7z^2-21z^3+14z^4}\\
\qquad\qquad\qquad\quad 15z^3-14z^4\\
\qquad\qquad\qquad\quad \underline{15z^3-45z^4+30z^5}\\
\qquad\qquad\qquad\qquad 31z^4-30z^5\\
\qquad\qquad\qquad\qquad \underline{31z^4-93z^5+62z^6}\\
\qquad\qquad\qquad\qquad\quad 63z^5-62z^6\\
\qquad\qquad\qquad\qquad\quad \underline{63z^5-189z^6+126z^7}\\
\qquad\qquad\qquad\qquad\qquad 127z^6-126z^7
\end{array}
$$

$$\therefore\ X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

$$= 2z^2+6z^3+14z^4+30z^5+62z^6+126z^7+......$$

Comparing with power series for anti-causal signal,

$$X(z) = \sum_{n=-\infty}^{-1} x[n]z^{-n}$$

$$= x[-1]z + x[-2]z^2 + x[-3]z^3 + x[-4]z^4 + ...$$

$$\therefore\ x[n] = \{......., 126, 62, 30, 14, 6, 2, 0, \underset{\uparrow}{0}\}$$

ii. By Partial Fraction

In this method, we split the given ZT (z-transform) into partial fractions, generally of first degree rational functions. Then comparing each of the partial fraction with standard z-transforms, we obtain the inverse z-transform (IZT) of the given function X(z).

Some Basic Partial Fractions Formula

S.N.	Form of the rational function	From of the partial function
i.	$\frac{px+q}{(x-a)(x-b)}$; $a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
ii.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{(x-a)}+\frac{B}{(x-a)^2}$
iii.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
iv.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

Example 2.4:

Find the inverse of $H(z) = \frac{(1+2z^{-1}+z^{-2})}{(1-0.75z^{-1}+0.125z^{-2})}$; ROC: $0.25 < |z| < 0.5$. **[2080 Bhadra]**

Solution:

Given, $H(z) = \frac{1+2z^{-1}+z^{-2}}{1-0.75z^{-1}+0.125z^{-2}}$; ROC $0.25 < |z| < 0.5$

$$H(z) = \frac{1+\frac{2}{z}+\frac{1}{z^2}}{1-\frac{0.75}{z}+\frac{0.125}{z^2}} = \frac{z^2+2z+1}{z^2-0.75z+0.125}$$

$$\text{or, } H(z) = \frac{z^2+2z+1}{(z-0.25)\,(z-0.5)}$$

$$\text{Let } F(z) = \frac{H(z)}{z} = \frac{z^2+2z+1}{z(z-0.25)(z-0.5)}$$

Using partial fraction, we have

$$F(z) = \frac{z^2+2z+1}{z(z-0.25)(z-0.5)} = \frac{A}{z}+\frac{B}{z-0.25}+\frac{C}{z-0.5}$$

$$= \frac{A(z-0.25)(z-0.5)+Bz(z-0.5)+Cz(z-0.25)}{z(z-0.25)(z-0.5)}$$

$$= \frac{A(z^2-0.75z+0.125)+Bz(z-0.5)+Cz(z-0.25)}{z(z-0.25)(z-0.5)}$$

$$= \frac{Az^2-0.75Az+0.125A+Bz^2-0.5Bz+Cz^2-0.25Cz}{z(z-0.25)(z-0.5)}$$

$$= \frac{(A+B+C)z^2-(0.75A+0.5B+0.25C)z+0.125A}{z(z-0.25)(z-0.5)}$$

Comparing coefficients, we get

$0.125A = 1$

or, $A = \frac{1}{0.125} = 8$

and,

$A + B + C = 1$

or, $8 + B + C = 1$

or, $B + C = -7$ .... (i)

and,

$-0.75A - 0.5B - 0.25C = 2$

or, $-0.5B - 0.25C = 2 + 0.75 \times 8$

or, $-0.5B - 0.25C = 8$ .... (ii)

On solving (i) and (ii), we get

$B = -25$ and $C = 18$

Therefore,

$$F(z) = \frac{8}{z} - \frac{25}{(z - 0.25)} + \frac{18}{(z - 0.5)}$$

$$\text{or, } \frac{H(z)}{z} = \frac{8}{z} - \frac{25}{(z - 0.25)} + \frac{18}{(z - 0.5)}$$

$$\text{or, } H(z) = \frac{8z}{z} - \frac{25z}{(z - 0.25)} + \frac{18z}{(z - 0.5)}$$

$$\text{or, } H(z) = 8 - \frac{25}{1 - 0.25z^{-1}} + \frac{18}{1 - 0.5z^{-1}}$$

Taking inverse z-transform, we get

$$h[n] = 8\delta[n] - 25(0.25)^n u[n] - 18(0.5)^n u[-n - 1]$$

Example 2.5:

Find the inverse z-transform of $X(z) = \frac{(2z^4 + 2z^3 - 3z + 2)}{(z^2 - 1.5z - 1)}$; ROC: $|z| < 0.5$, using partial fraction expansion method.

Solution: **[2079 Bhadra]**

Given, $X(z) = \frac{2z^4 + 2z^3 - 3z + 2}{z^2 - 1.5z - 1}$; ROC: $|z| < 0.5$

Using long division method, we have

$$\begin{array}{r|l} z^2-1.5z-1 & 2z^4+2z^3-3z+2 \;(2z^2+5z \\ & \underline{2z^4-3z^3-2z^2} \\ & \quad 5z^3+2z^2-3z+2 \\ & \quad \underline{5z^3-7.5z^2-5z} \\ & \qquad 9.5z^2+2z+2 \end{array}$$

$$\text{So, } X(z) = 2z^2 + 5z + \frac{9.5z^2 + 2z + 2}{z^2 - 1.5z - 1}$$

$$\text{Let } F(z) = \frac{X(z)}{z} = 2z + 5 + \frac{9.5z^2 + 2z + 2}{z(z^2 - 1.5z - 1)}$$

$$\text{Suppose } Y(z) = \frac{9.5z^2 + 2z + 2}{z(z^2 - 1.5z - 1)} = \frac{9.5z^2 + 2z + 2}{z(z + 0.5)(z - 2)}$$

Using partial fraction, we have

$$Y(z) = \frac{A}{z} + \frac{B}{z + 0.5} + \frac{C}{z - 2}$$

$$= \frac{A(z + 0.5)(z - 2) + Bz(z - 2) + Cz(z + 0.5)}{z(z + 0.5)(z - 2)}$$

$$= \frac{A(z^2 - 1.5z - 1) + Bz(z - 2) + Cz(z + 0.5)}{z(z + 0.5)(z - 2)}$$

$$= \frac{Az^2 - 1.5Az - A + Bz^2 - 2Bz + Cz^2 + 0.5Cz}{z(z + 0.5)(z - 2)}$$

$$= \frac{(A + B + C)z^2 - (1.5A + 2B - 0.5C)z - A}{z(z + 0.5)(z - 2)}$$

Comparing coefficients, we get

$-A = 2$

or, $A = -2$

and,

$-1.5A - 2B + 0.5C = 2$

or, $-2B + 0.5C = 2 + 1.5 \times -2$

or, $-2B + 0.5C = -1$ ....(i)

and,

$A + B + C = 9.5$

or, $B + C = 9.5 - (-2)$

or, $B + C = 11.5$ .... (ii)

On solving (i) and (ii), we get

$B = 2.7$ and $C = 8.8$

Therefore,

$$Y(z) = \frac{-2}{z} + \frac{2.7}{z + 0.5} + \frac{8.8}{z - 2}$$

and,

$$F(z) = \frac{X(z)}{z} = 2z + 5 - \frac{2}{z} + \frac{2.7}{z + 0.5} + \frac{8.8}{z - 2}$$

$$\text{or, } X(z) = 2z^2 + 5z - 2 + \frac{2.7z}{z + 0.5} + \frac{8.8z}{z - 2}$$

$$\text{or, } X(z) = 2z^2 + 5z - 2 + \frac{2.7}{1 + 0.5z^{-1}} + \frac{8.8}{1 - 2z^{-1}}$$

Taking inverse z-transform, we get

$$x[n] = 2\delta[n + 2] + 5\delta[n + 1] - 2\delta[n] - 2.7(-0.5)^n u(-n - 1] - 8.8(2)^n u[-n - 1]$$

Example 2.6:

Find the inverse z-transform for $H(z) = \frac{z}{3z^2 - 4z + 1}$ using partial fraction method for $\frac{1}{3} < |z| < 1$. [2079 Baishakh]

Solution:

Given, $H(z) = \frac{z}{3z^2 - 4z + 1}$; ROC: $\frac{1}{3} < |z| < 1$

$$\text{or, } H(z) = \frac{z}{3(z - 1)\left(z - \frac{1}{3}\right)}$$

$$\text{Let, } F(z) = \frac{H(z)}{z} = \frac{1}{3(z - 1)\left(z - \frac{1}{3}\right)}$$

Using partial fraction, we have

$$F(z) = \frac{1}{3(z - 1)\left(z - \frac{1}{3}\right)} = \frac{A}{3(z - 1)} + \frac{B}{\left(z - \frac{1}{3}\right)}$$

$$= \frac{A\left(z - \frac{1}{3}\right) + 3B(z - 1)}{3(z - 1)\left(z - \frac{1}{3}\right)}$$

$$= \frac{Az - \frac{1}{3}A + 3Bz - 3B}{3(z - 1)\left(z - \frac{1}{3}\right)}$$

$$= \frac{(A + 3B)z - \frac{1}{3}A - 3B}{3(z - 1)\left(z - \frac{1}{3}\right)}$$

Comparing coefficients, we get

$A + 3B = 0 \Rightarrow A = -3B$

and,

$-\frac{1}{3}A - 3B = 1$

or, $-\frac{1}{3} \times -3B - 3B = 1$

or, $B - 3B = 1$

or, $-2B = 1$

or, $B = -\frac{1}{2}$

So, $A = -3B = -3 \times \frac{-1}{2} = \frac{3}{2}$

Therefore,

$$F(z) = \frac{H(z)}{z} = \frac{3/2}{3(z - 1)} - \frac{1/2}{\left(z - \frac{1}{3}\right)}$$

or, $\frac{H(z)}{z} = \frac{1/2}{z-1} - \frac{1/2}{z-\frac{1}{3}}$

or, $H(z) = \frac{\frac{1}{2}z}{z-1} - \frac{\frac{1}{2}z}{z-\frac{1}{3}}$

or, $H(z) = \frac{1/2}{1-z^{-1}} - \frac{1/2}{1-\frac{1}{3}z^{-1}}$

Taking inverse z-transform, we get

$h[n] = \frac{-1}{2}(1)^n u[-n-1] - \frac{1}{2}\left(\frac{1}{3}\right)^n u[n]$

Example 2.7:

Find the inverse z-transform of $X(z) = \frac{(z^3+z^2+1.5z+0.5)}{(z^3+1.5z^2+0.5z)}$; ROC: $|z| < \frac{1}{2}$. [2078 Bhadra]

Solution:

Given, $X(z) = \frac{z^3+z^2+1.5z+0.5}{z^3+1.5z^2+0.5z}$; ROC: $|z| < \frac{1}{2}$

Let, $F(z) = \frac{X(z)}{z} = \frac{z^3+z^2+1.5z+0.5}{z(z^3+1.5z^2+0.5z)} = \frac{z^3+z^2+1.5z+0.5}{z^2(z+1)(z+0.5)}$

Using partial fraction, we have

$$F(z) = \frac{z^3+z^2+1.5z+0.5}{z^2(z+1)(z+0.5)} = \frac{A}{z+1} + \frac{B}{z+0.5} + \frac{Cz+D}{z^2}$$

$$= \frac{Az^2(z+0.5)+Bz^2(z+1)+(Cz+D)(z+1)(z+0.5)}{z^2(z+1)(z+0.5)}$$

$$= \frac{Az^3+0.5Az^2+Bz^3+Bz^2+(Cz+D)(z^2+1.5z+0.5)}{z^2(z+1)(z+0.5)}$$

$$= \frac{Az^3+0.5Az^2+Bz^3+Bz^2+Cz^3+1.5Cz^2+0.5Cz+Dz^2+1.5Dz+0.5D}{z^2(z+1)(z+0.5)}$$

$$= \frac{(A+B+C)z^3+(0.5A+B+1.5C+D)z^2+(0.5C+1.5D)z+0.5D}{z^2(z+1)(z+0.5)}$$

Comparing coefficients, we get

$0.5D = 0.5$

$\Rightarrow D = 1$

and, $0.5C + 1.5D = 1.5$

or, $0.5C + 1.5 = 1.5$

$\Rightarrow C = 0$

and,

$0.5A + B + 1.5C + D = 1$

or, $0.5A + B + 1 = 1$

or, $0.5A + B = 0$ ....(i)

and,

$A + B + C = 1$

or, $A + B + 0 = 1$

or, $A + B = 1$ ....(ii)

On solving (i) and (ii), we get

$A = 2$ and $B = -1$

Therefore,

$F(z) = \frac{X(z)}{z} = \frac{2}{z+1} - \frac{1}{z+0.5} + \frac{1}{z^2}$

or, $X(z) = \frac{2z}{z+1} - \frac{z}{z+0.5} + \frac{z}{z^2}$

or, $X(z) = \frac{2}{1+z^{-1}} - \frac{1}{1+0.5z^{-1}} + \frac{1}{z}$

Taking inverse z-transform, we get

$x[n] = -2(-1)^n u[-n-1] + (-0.5)^n u[-n-1] + \delta[n-1]$

Example 2.8:

Find inverse z-transform of

$X(z) = \frac{z}{\{(z-0.6)(z+0.5)^2\}}$; ROC: $|z| > 0.6$ [2076 Ashwin]

Solution:

Given, $X(z) = \frac{z}{(z-0.6)(z+0.5)^2}$; ROC: $|z| > 0.6$

Let $F(z) = \frac{X(z)}{z} = \frac{1}{(z-0.6)(z+0.5)^2}$

Using partial fraction, we have

$$F(z) = \frac{1}{(z-0.6)(z+0.5)^2} = \frac{A}{(z-0.6)} + \frac{B}{(z+0.5)} + \frac{C}{(z+0.5)^2}$$

$$= \frac{A(z+0.5)^2 + B(z+0.5)(z-0.6) + C(z-0.6)}{(z-0.6)\,(z+0.5)^2}$$

$$= \frac{A(z^2+z+0.25) + B(z^2-0.1z-0.3) + C(z-0.6)}{(z-0.6)(z+0.5)^2}$$

$$= \frac{Az^2 + Az + 0.25A + Bz^2 - 0.1Bz - 0.3B + Cz - 0.6C}{(z-0.6)(z+0.5)^2}$$

$$= \frac{(A+B)z^2 + (A-0.1B+C)z + 0.25A - 0.3B - 0.6C}{(z-0.6)\,(z+0.5)^2}$$

Comparing coefficients, we get

$A + B = 0$ .... (i)

$A - 0.1B + C = 0$ .... (ii)

$0.25A - 0.3B - 0.6C = 1$ .... (iii)

On solving (i), (ii) and (iii), we get

$A = 0.826,\ B = -0.826,\ C = -0.909$

Therefore,

$$F(z) = \frac{X(z)}{z} = \frac{0.826}{(z-0.6)} - \frac{0.826}{(z+0.5)} - \frac{0.909}{(z+0.5)^2}$$

$$\text{or, } X(z) = \frac{0.826z}{(z-0.6)} - \frac{0.826z}{(z+0.5)} - \frac{0.909z}{(z+0.5)^2}$$

$$\text{or, } X(z) = \frac{0.826}{1-0.6z^{-1}} - \frac{0.826}{1+0.5z^{-1}} + \frac{0.909}{0.5} \times \frac{-0.5z}{(z+0.5)^2}$$

Taking inverse z-transform, we get

$$x[n] = 0.826(0.6)^n u[n] - 0.826(-0.5)^n u[n] + 1.818n(-0.5)^n u[n]$$

Example 2.9:

Find the inverse z-transform of $X(z) = \frac{(2z^3+2z^2+3z+5)}{(z^2-0.1z-0.2)}$; ROC: $|z| < 0.4$. [2073 Chaitra]

Solution:

Given, $X(z) = \frac{2z^3+2z^2+3z+5}{z^2-0.1z-0.2}$; ROC: $|z| < 0.4$

Using long division method, we have

$$\begin{array}{r} z^2-0.1z-0.2\,\overline{)\,2z^3+2z^2+3z+5\,}(\,2z \\ \underline{2z^3-0.2z^2-0.4z} \\ 2.2z^2+3.4z+5 \end{array}$$

So, $X(z) = 2z + \frac{2.2z^2+3.4z+5}{z^2-0.1z-0.2}$

Let $F(z) = \frac{X(z)}{z} = 2 + \frac{2.2z^2+3.4z+5}{z(z^2-0.1z-0.2)}$

Suppose $Y(z) = \frac{2.2z^2+3.4z+5}{z(z^2-0.1z-0.2)} = \frac{(2.2z^2+3.4z+5)}{z(z+0.4)\,(z-0.5)}$

Using partial fraction, we have

$$Y(z) = \frac{2.2z^2+3.4z+5}{z(z+0.4)\,(z-0.5)} = \frac{A}{z} + \frac{B}{z+0.4} + \frac{C}{z-0.5}$$

$$= \frac{A(z+0.4)(z-0.5) + Bz(z-0.5) + Cz(z+0.4)}{z(z+0.4)(z-0.5)}$$

$$= \frac{A(z^2-0.1z-0.2) + Bz(z-0.5) + Cz(z+0.4)}{z(z+0.4)(z-0.5)}$$

$$= \frac{Az^2 - 0.1Az - 0.2A + Bz^2 - 0.5Bz + Cz^2 + 0.4Cz}{z(z+0.4)(z-0.5)}$$

$$= \frac{(A+B+C)z^2 - (0.1A+0.5B-0.4C)z - 0.2A}{z(z+0.4)\,(z-0.5)}$$

Comparing coefficients, we get

$-0.2A = 5$

$\Rightarrow A = -25$

and,

$-0.1A - 0.5B + 0.4C = 3.4$

or, $-0.5B + 0.4C = 3.4 + 0.1 \times -25$

or, $-0.5B + 0.4C = 0.9$ .... (i)

and,

$A + B + C = 2.2$

or, $B + C = 2.2 - (-25)$

or, $B + C = 27.2$ .... (ii)

On solving (i) and (ii), we get

$B = 11.088$ and $C = 16.111$

Therefore,

$$Y(z) = \frac{-25}{z} + \frac{11.088}{z+0.4} + \frac{16.111}{z-0.5}$$

and,

$$F(z) = \frac{X(z)}{z} = 2 - \frac{25}{z} + \frac{11.088}{z+0.4} + \frac{16.111}{z-0.5}$$

$$\text{or, } X(z) = 2z - \frac{25z}{z} + \frac{11.088z}{z+0.4} + \frac{16.111z}{z-0.5}$$

$$= 2z - 25 + \frac{11.088}{1+0.4z^{-1}} + \frac{16.111}{1-0.5z^{-1}}$$

Taking inverse z-transform, we get

$$x[n] = 2\delta[n+1] - 25\delta[n] - 11.088(-0.4)^n u[-n-1] - 16.111\,(0.5)^n u[-n-1]$$

Note:

- We take $\frac{X(z)}{z}$ form since it is easier to take inverse z-transform.
- For $H(z)$ or $X(z) = \frac{N(z)}{D(z)}$ (Power of z or order of z)

i. If $N(z) < D(z)$, take $\frac{X(z)}{z}$ form

ii. If $N(z) = D(z)$, take $\frac{X(z)}{z}$ form

iii. If $N(z) > D(z)$, perform long division till $N(z) = D(z)$ or $N(z) < D(z)$, then take $\frac{X(z)}{z}$ form.

Chapter – 3

ANALYSIS OF LTI SYSTEM IN FREQUENCY DOMAIN

3.1 Frequency Response of LTI System, Response to Complex Exponential

Please refer to Chapter 1 (1.9) for this.

Examples 3.1:

Plot the pole-zero on the z-plane and draw magnitude response (not to the scale) of an LTI system described by the equation, y(n) = x(n) + 0.8x(n – 1) + 0.8x(n – 2) + 0.49y(n – 2). **[2080 Bhadra]**

Solution:

10 Marks //

Given equation is,

$y(n) = x(n) + 0.8x(n-1) + 0.8x(n-2) + 0.49y(n-2)$

or, $y(n) - 0.49y(n-2) = x(n) + 0.8x(n-1) + 0.8x(n-2)$

Taking z-transform, we get

$Y(z) - 0.49z^{-2}Y(z) = X(z) + 0.8z^{-1}X(z) + 0.8z^{-2}X(z)$

or, $(1 - 0.49z^{-2})Y(z) = (1 + 0.8z^{-1} + 0.8z^{-2})X(z)$

$$\text{or, } \frac{Y(z)}{X(z)} = \frac{1 + 0.8z^{-1} + 0.8z^{-2}}{1 - 0.49z^{-2}}$$

$$\text{or, } H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{0.8}{z} + \frac{0.8}{z^2}}{1 - \frac{0.49}{z^2}} = \frac{\frac{z^2 + 0.8z + 0.8}{z^2}}{\frac{z^2 - 0.49}{z^2}}$$

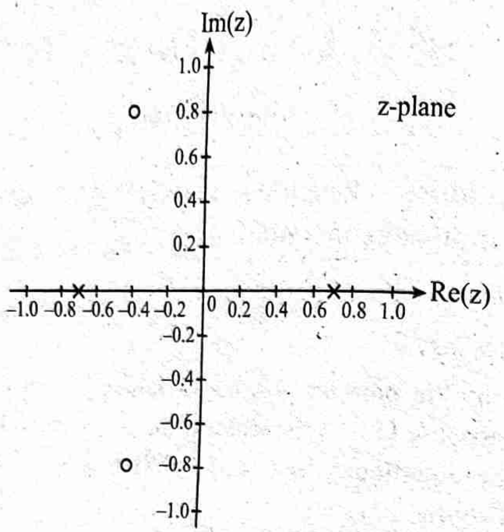
$$\therefore\ H(z) = \frac{z^2 + 0.8z + 0.8}{z^2 - 0.49}$$

Here,

Poles: $z^2 - 0.49 = 0 \Rightarrow z = \pm 0.7$

Zeros: $z^2 + 0.8z + 0.8 \Rightarrow z = -0.4 + j0.8, -0.4 - j0.8$

Plotting poles and zeros, we have



Put $z = e^{j\omega}$,

$$H(e^{j\omega}) = \frac{(e^{j\omega})^2 + 0.8(e^{j\omega}) + 0.8}{(e^{j\omega})^2 - 0.49}$$

$$= \frac{e^{j2\omega} + 0.8e^{j\omega} + 0.8}{e^{j2\omega} - 0.49}$$

We have,

$$N(e^{j\omega}) = e^{j2\omega} + 0.8e^{j\omega} + 0.8$$

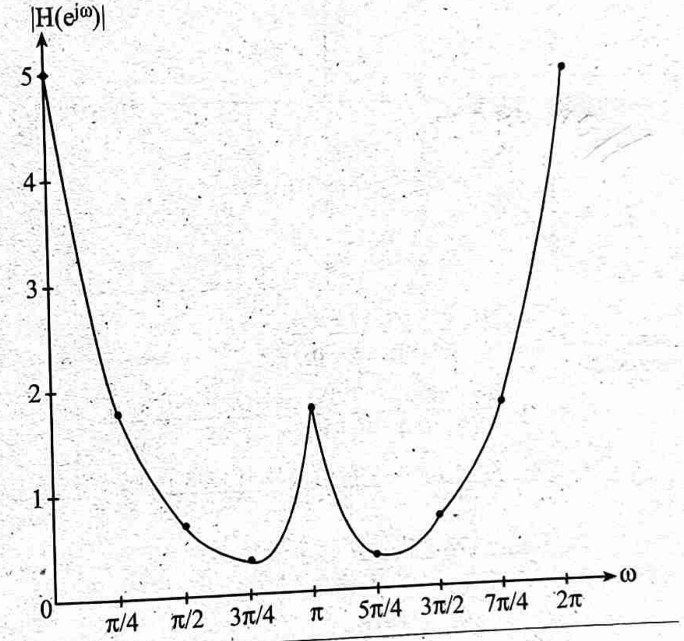
$$= \cos(2\omega) + j\sin(2\omega) + 0.8\cos(\omega) + j0.8\sin(\omega) + 0.8$$

and, $D(e^{j\omega}) = e^{j2\omega} - 0.49 = \cos(2\omega) + j\sin(2\omega) - 0.49$

Now,

ω	$N(e^{j\omega})$	$D(e^{j\omega})$	$H(e^{j\omega})$	$\lvert H(e^{j\omega})\rvert$
0	2.6	0.51	5.098	5.098
$\frac{\pi}{4}$	1.36568 + j1.56568	−0.49 + j	0.723 – j1.72	1.864
$\frac{\pi}{2}$	−0.2 + j0.8	−1.49	0.134 – j0.537	0.5534
$\frac{3\pi}{4}$	0.2343 – j0.4343	−0.49 – j	0.2576 + j0.36	0.4427
π	1	0.51	1.96	1.96
$\frac{5\pi}{4}$	0.2343 + j0.4343	−0.49 + j	0.2576 – j0.36	0.4427
$\frac{3\pi}{2}$	−0.2 – j0.8	−1.49	0.134 + j0.537	0.5534
$\frac{7\pi}{4}$	1.36568 – j1.56568	−0.49 – j	0.723 + j1.72	1.864
2π	2.6	0.51	5.098	5.098

Magnitude response:



Example 3.2:

The poles of a system are located at 0.45 ± j1.6 and zeros at 0.58 ± j2.06. Map the poles and zeros in the z-plane and plot the magnitude response (not in scale) of the system. [080 Baishakh, 076 Chaitra, 075 Ashwin]

Solution:

Given,

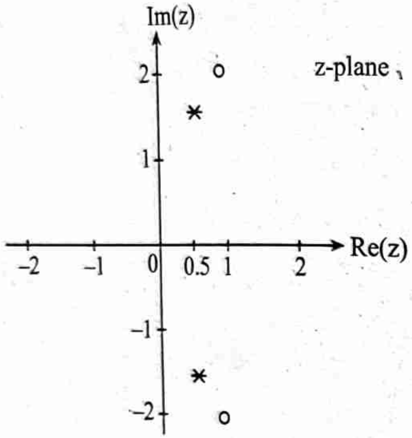
Poles: $p_1 = 0.45 + j1.6$

$p_2 = 0.45 - j1.6$

Zeros: $z_1 = 0.58 + j2.06$

$z_2 = 0.58 - j2.06$

Plotting poles and zeros, we have



$$H(z) = \frac{N(z)}{D(z)} = \frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)}$$

$$\text{or, } H(z) = \frac{(z - 0.58 - j2.06)(z - 0.58 + j2.06)}{(z - 0.45 - j1.6)(z - 0.45 + j1.6)}$$

$$= \frac{z^2 – 0.58z + j2.06z – 0.58z + 0.3364 – j1.1948 – j2.06z + j1.1948 – j^2 4.2436}{z^2 – 0.45z + j1.6z – 0.45z + 0.2025 – j0.72 – j1.6z + j0.72 – j^2 2.56}$$

$$= \frac{z^2 - 1.16z + 0.3364 + 4.2436}{z^2 - 0.9z + 0.2025 + 2.56}$$

$$\therefore \; H(z) = \frac{z^2 - 1.16z + 4.58}{z^2 - 0.9z + 2.7625}$$

Put $z = e^{j\omega}$,

$$H(e^{j\omega}) = \frac{(e^{j\omega})^2 - 1.16(e^{j\omega}) + 4.58}{(e^{j\omega})^2 - 0.9(e^{j\omega}) + 2.7625}$$

$$= \frac{e^{j2\omega} - 1.16e^{j\omega} + 4.58}{e^{j2\omega} - 0.9e^{j\omega} + 2.7625}$$

We have,

$$N(e^{j\omega}) = e^{j2\omega} - 1.16e^{j\omega} + 4.58$$

$$= \cos(2\omega) + j\sin(2\omega) – 1.16\cos(\omega) – j1.16\sin(\omega) + 4.58$$

and,

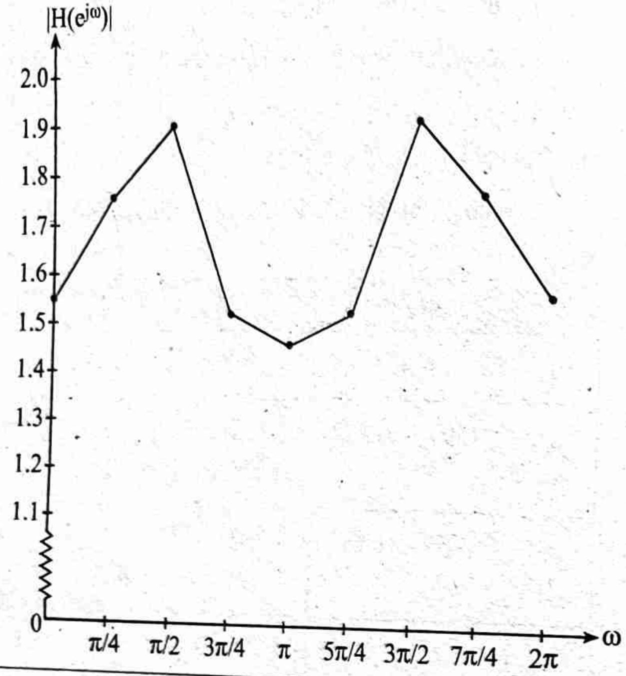
$$D(e^{j\omega}) = e^{j2\omega} - 0.9e^{j\omega} + 2.7625$$

$$= \cos(2\omega) + j\sin(2\omega) – 0.9\cos(\omega) – j0.9\sin(\omega) + 2.7625$$

Now,

ω	**$N(e^{j\omega})$**	**$D(e^{j\omega})$**	**$H(e^{j\omega})$**	**$\lvert H(e^{j\omega})\rvert$**
0	4.42	2.8625	1.5441	1.5441
$\frac{\pi}{4}$	3.7597 + j0.1797	2.1261 + j0.3636	1.7321 – j0.2117	1.7449
$\frac{\pi}{2}$	3.58 – j1.16	1.7625 – j0.9	1.8776 + j0.3	1.9014
$\frac{3\pi}{4}$	5.4 – j1.82	3.3988 – j1.6363	1.499 + j0.186	1.5105
π	6.74	4.6625	1.44557	1.4456
$\frac{5\pi}{4}$	5.4 + j1.82	3.3988 + j1.6363	1.499 – j0.186	1.5105
$\frac{3\pi}{2}$	3.58 + j1.16	1.7625 + j0.9	1.8776 – j0.3	1.9014
$\frac{7\pi}{4}$	3.7597 – j0.1797	2.1261 – j0.3636	1.7321 + j0.2117	1.7449
2π	4.42	2.8625	1.5441	1.5441

Magnitude response:



Example 3.3:

Plot the pole-zero in z-plane and draw the magnitude response (not to the scale) of the equation of the system described by difference equation.

$$y[n] - 0.35y[n-1] + 0.25y[n-2] = x[n] - 0.75x[n-1].$$

[2079 Bhadra]

Solution:

Given equation is,

$$y[n] - 0.35y[n-1] + 0.25y[n-2]$$
$$= x[n] - 0.75x[n-1]$$

Taking z-transform, we get

$$Y(z) - 0.35z^{-1}Y(z) + 0.25z^{-2}Y(z) = X(z) - 0.75z^{-1}X(z)$$

or, $(1 - 0.35z^{-1} + 0.25z^{-2})Y(z) = (1 - 0.75z^{-1})X(z)$

or, $\frac{Y(z)}{X(z)} = \frac{(1 - 0.75z^{-1})}{(1 - 0.35z^{-1} + 0.25z^{-2})}$

or, $H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{0.75}{z}}{1 - \frac{0.35}{z} + \frac{0.25}{z^2}} = \frac{\frac{z - 0.75}{z}}{\frac{z^2 - 0.35z + 0.25}{z^2}}$

$$= \frac{z - 0.75}{z} \times \frac{z^2}{z^2 - 0.35z + 0.25}$$

$$= \frac{z(z - 0.75)}{z^2 - 0.35z + 0.25}$$

$\therefore\ H(z) = \frac{z^2 - 0.75z}{z^2 - 0.35z + 0.25}$

Here,

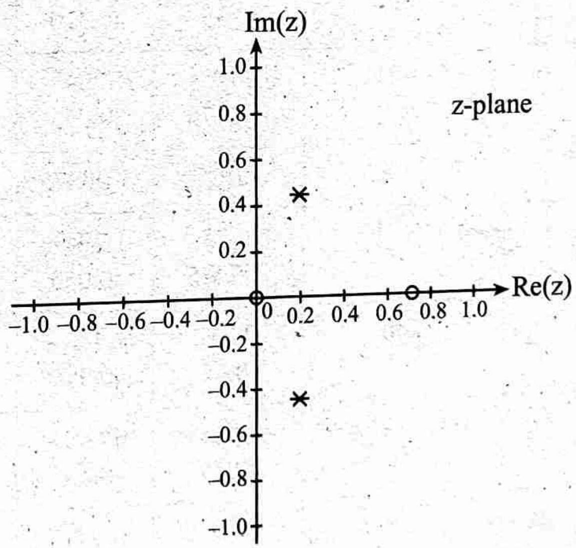
Poles: $z^2 - 0.35z + 0.25 = 0$

$\Rightarrow z = 0.175 + j0.468, 0.175 - j0.468$

Zeros: $z^2 - 0.75z = 0$

$\Rightarrow z = 0, 0.75$

Plotting poles and zeros, we have



Put $z = e^{j\omega}$,

$$H(e^{j\omega}) = \frac{(e^{j\omega})^2 - 0.75(e^{j\omega})}{(e^{j\omega})^2 - 0.35(e^{j\omega}) + 0.25}$$

$$= \frac{e^{j2\omega} - 0.75e^{j\omega}}{e^{j2\omega} - 0.35e^{j\omega} + 0.25}$$

We have,

$$N(e^{j\omega}) = e^{j2\omega} - 0.75e^{j\omega}$$

$$= \cos(2\omega) + j\sin(2\omega) - 0.75\cos(\omega) - 0.75\sin(\omega)$$

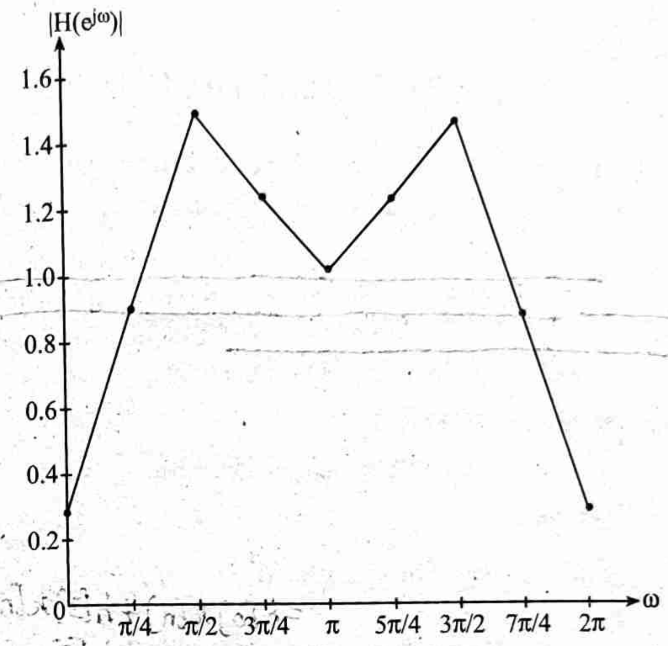
and,

$$D(e^{j\omega}) = e^{j2\omega} - 0.35e^{j\omega} + 0.25$$

$$= \cos(2\omega) + j\sin(2\omega) - 0.35\cos(\omega) - j0.35\sin(\omega) + 0.25$$

Now,

ω	$N(e^{j\omega})$	$D(e^{j\omega})$	$H(e^{j\omega})$	$\lvert H(e^{j\omega})\rvert$
0	0.25	0.9	0.2777	0.2777
$\frac{\pi}{4}$	–0.53 + j0.469	–0.025 + j0.75	0.648 + j0.685	0.943
$\frac{\pi}{2}$	–1 – j0.75	–0.75 – j0.35	1.478 + j0.310	1.51
$\frac{3\pi}{4}$	0.53 – j1.53	0.4975 – j1.2475	1.204 – j0.055	1.205
π	1.75	1.6	1.09375	1.09375
$\frac{5\pi}{4}$	0.53 + j1.53	0.4975 + j1.2475	1.204 + j0.055	1.205
$\frac{3\pi}{2}$	–1 + j0.75	–0.75 + j0.35	1.478 – j0.310	1.51
$\frac{7\pi}{4}$	–0.53 – j0.469	–0.025 – j0.75	0.648 – j0.685	0.943
2π	0.25	0.9	0.2777	0.2777



Chapter – 4

DISCRETE FILTER STRUCTURES

4.1 FIR filter, Structures for FIR Filter (Direct form, Cascade, Frequency Sampling, Lattice)

FIR filter is a type of digital filter that has finite impulse response, meaning that the output of the filter is solely determined by a finite number of past input samples. FIR filters are employed where there is a requirement of linear phase in pass band.

Structures for FIR Filter

In general, an FIR system is described by the difference equation

$$y[n] = \sum_{k=0}^{m-1} h[k]\, x[n-k] \text{ .... (i)}$$

$$= h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] \overline{+} h[m-1]x[n-m+1]$$

Taking z-transform, we have

$$Y(z) = \sum_{k=0}^{m-1} h[k]\, z^{-k}\, X(z)$$

$$\text{or, } Y(z) = h(0)X(z) + h(1)z^{-1}X(z) + h(2)z^{-2}X(z) + \text{......} + h(m-1)z^{-(m-1)}X(z)$$

$$= [h(0) + h(1)z^{-1} + h(2)z^{-2} + \text{....} + h(m-1)z^{-(m-1)}]X(z)$$

$$\text{or, } H(z) = \frac{Y(z)}{X(z)}$$

$$= h[0] + h[1]z^{-1} + h[2]z^{-2} + \text{....} + h[m-1]z^{-(m-1)}$$

$$\therefore H(z) = \sum_{k=0}^{m-1} h[k]\, z^{-k} \text{ ...... (ii)}$$

Direct Form Structure

From equation (i),

$$y[n] = \sum_{k=0}^{m-1} h[k]\, x[n-k]$$

$$= h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] + \text{...} + h[m-1]x[n-m+1]$$

The structure is illustrated below:

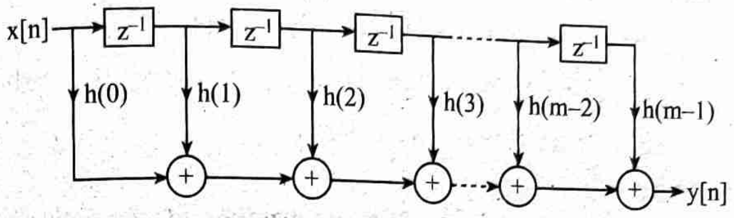


Fig.: Direct form realization of FIR system

Computation Complexity and Memory Requirement

Computational complexity refers to the number of arithmetic operations (multiplications, divisions, and additions) required to compute an output value y(n) for the system.

- FIR system has a complexity of m multiplications and m–1 additions.

Memory requirement refers to the number of memory locations required to store the system parameters, past inputs, past outputs and any intermediate computed values.

- FIR system requires m–1 memory locations for storage and m–1 delay blocks.

Example 4.1:

Determine the direct form realization of the following system function.

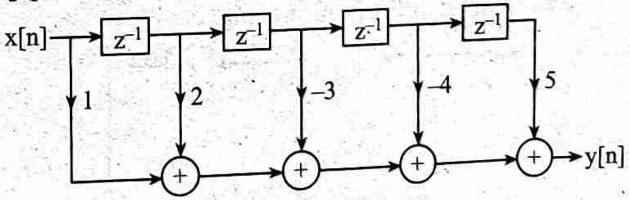
$$\mathbf{H(z) = 1 + 2z^{-1} - 3z^{-2} - 4z^{-3} + 5z^{-4}}$$

$$a_3^{(4)} = a_2^{(3)} = \frac{a_2^4 + k_4 a_2^4}{1 - k_4^2}$$

Solution:

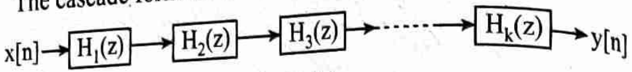
Comparing above equation with $H(z) = \sum_{k=0}^{m-1} h[k]z^{-k}$, we have

$h[0] = 1, h[1] = 2, h[2] = -3, h[3] = -4, h[4] = 5$



Cascade Form Structure

The cascade form for FIR filter can be represented as:



$H(z) = H_1(z)H_2(z)H_3(z)$ ...... $H_k(z)$

Example 4.2:

Obtain the cascade form realization of the following function: $H(z) = (1 + 2z^{-1} - z^{-2})\,(1 + z^{-1} - z^{-2})$

Solution:

Given system function is

$H(z) = (1 + 2z^{-1} - z^{-2})\,(1 + z^{-1} - z^{-2}) = H_1(z)\,H_2(z)$

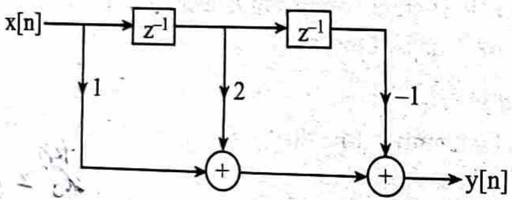
For $H_1(z) = 1 + 2z^{-1} - z^{-2}$,

Comparing with general system function of FIR filter, H(z)

$= \sum_{k=0}^{m-1} h[k]\,z^{-k}$, we get

$h[0] = 1, h[1] = 2, h[2] = -1$

The required direct form structure is



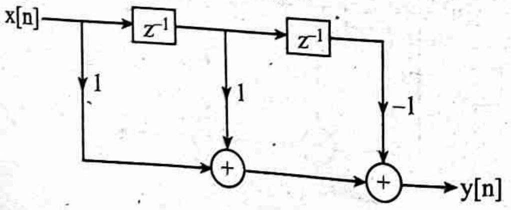
For $H_2(z) = 1 + z^{-1} - z^{-2}$,

Comparing with general system function of FIR filter, H(z)

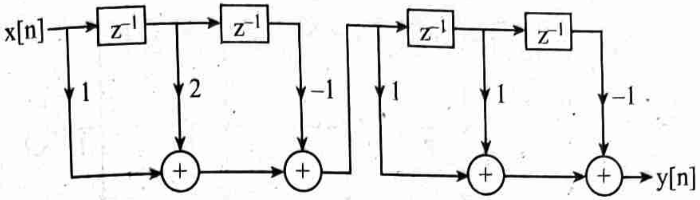
$= \sum_{k=0}^{m-1} h[k]\,z^{-k}$, we get

$h[0] = 1, h[1] = 1, h[2] = -1$

The required direct form structure is



Therefore, the overall structure is



Example 4.3:

Determine direct form and cascade form realization of $H(z) = \left(1 - \frac{1}{4}z^{-1} + \frac{3}{8}z^{-2}\right)\left(1 - \frac{1}{8}z^{-1} - \frac{1}{2}z^{-2}\right)$.

Solution:

Given system function is

$$H(z) = \left(1 - \frac{1}{4}z^{-1} + \frac{3}{8}z^{-2}\right)\left(1 - \frac{1}{8}z^{-1} - \frac{1}{2}z^{-2}\right)$$

$$= 1\left(1 - \frac{1}{8}z^{-1} - \frac{1}{2}z^{-2}\right) - \frac{1}{4}z^{-1}\left(1 - \frac{1}{8}z^{-1} - \frac{1}{2}z^{-2}\right) +$$

$$\frac{3}{8}z^{-2}\left(1 - \frac{1}{8}z^{-1} - \frac{1}{2}z^{-2}\right)$$

$$= 1 - \frac{1}{8}z^{-1} - \frac{1}{2}z^{-2} - \frac{1}{4}z^{-1} + \frac{1}{32}z^{-2} + \frac{1}{8}z^{-3} + \frac{3}{8}z^{-2} - \frac{3}{64}z^{-3}$$

$$- \frac{3}{16}z^{-4}$$

or, $H(z) = 1 - \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{5}{64}z^{-3} + \frac{3}{16}z^{-4}$ .... (ii)

The general system function of a FIR filter is

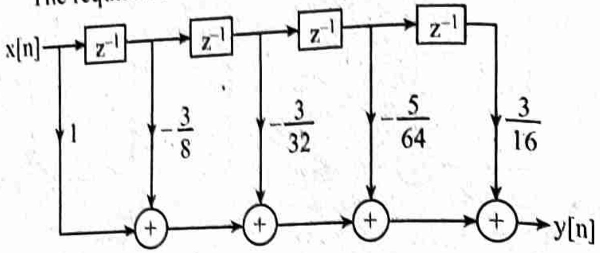
$H(z) = \sum_{k=0}^{m-1} h[k]z^{-k}$ ....(iii)

i. **Direct form:**

Comparing equation (ii) with equation (iii),

$h[0] = 1, h[1] = -\frac{3}{8}, h[2] = -\frac{3}{32}, h[3] = -\frac{5}{64}, h[4] = \frac{3}{16}$

The required direct form structure is



ii. Cascade form:

From equation (i),

$$H(z) = \left(1 - \frac{1}{4}z^{-1} + \frac{3}{8}z^{-2}\right)\left(1 - \frac{1}{8}z^{-1} - \frac{1}{2}z^{-2}\right)$$

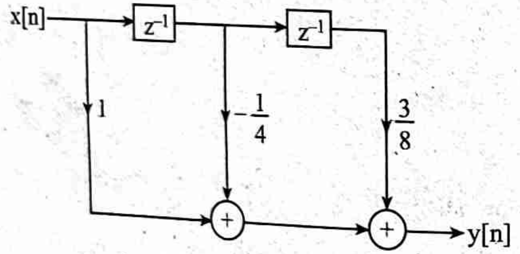
$$= H_1(z)\, H_2(z)$$

For $H_1(z) = \left(1 - \frac{1}{4}z^{-1} + \frac{3}{8}z^{-2}\right)$,

Comparing with equation (iii), we get

$h[0] = 1, h[1] = -\frac{1}{4}, h[2] = \frac{3}{8}$

The required direct from structure is



For $H_2(z) = \left(1 - \frac{1}{8}z^{-1} - \frac{1}{2}z^{-2}\right)$,

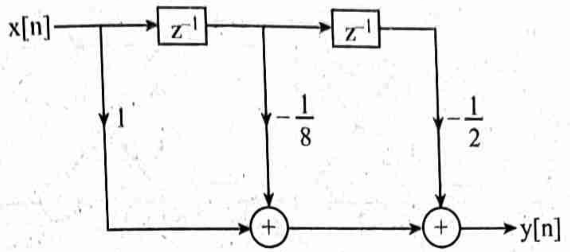
Comparing with equation (iii), we get

$h[0] = 1$,

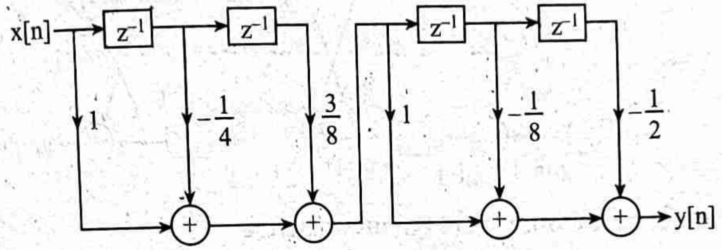
$h[1] = -\frac{1}{8}$,

$h[2] = -\frac{1}{2}$

The required direct form structure is



Therefore, the overall structure is



Lattice Structure

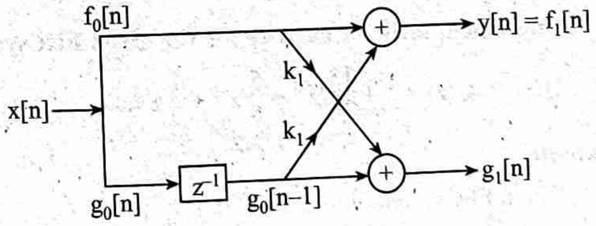
Lattice filters are used extensively in digital speech processing and in the implementation of adaptive filters.

Let us consider a FIR filter with system function

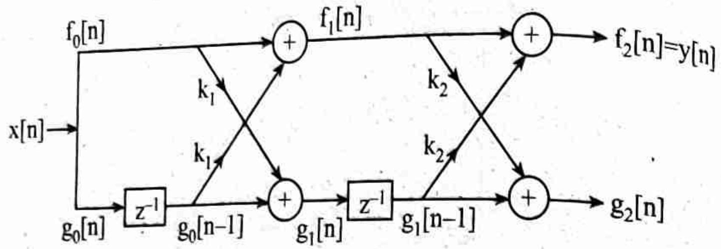
$$H(z) = A_m(z)$$

$$= 1 + \sum_{k=1}^{m} a_m(k)\, z^{-k} \; ; m \geq 1 \text{ and } A_0(z) = 1$$

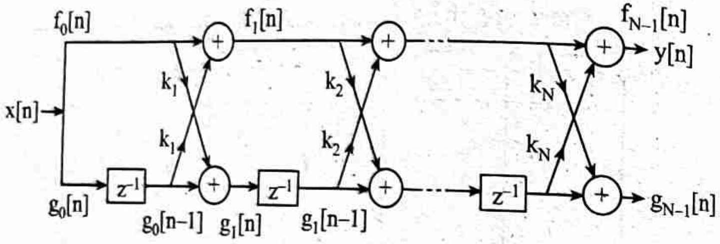
Single stage lattice:



Two stage lattice:



N-stage lattice:



Conversion from Direct Form to Lattice

$a_{m-1}(0) = 1$

$a_{m-1}(k) = \dfrac{a_m(k) - a_m(m)\, a_m(m-k)}{1 - a_m^2(m)}$; $1 \le k \le m-1$

$k_m = a_m(m)$

Conversion from Lattice to Direct Form

$\alpha_m(0) = 1$

$\alpha_m(m) = k_m$

$\alpha_m(k) = \alpha_{m-1}(k) + \alpha_m(m)\alpha_{m-1}(m-k)$

Example 4.4:

Implement lattice structure for the given FIR system.

$$H(z) = A_3(z) = 1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3}.$$

Solution:

Given FIR system is

$$H(z) = 1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3} \text{ .... (i)}$$

Order of the system = 3

Comparing equation (i) with general system function of causal FIR system, $H(z) = 1 + \sum_{k=1}^{m} a_m(k)z^{-k}$, we have

$$a_3(1) = \frac{13}{24},\ a_3(2) = \frac{5}{8},\ a_3(3) = \frac{1}{3}$$

Also,

$$k_1 = a_1(1) = ?,\ k_2 = a_2(2) = ?,\ k_3 = a_3(3) = \frac{1}{3}$$

Now, using $a_{m-1}(k) = \dfrac{a_m(k) - a_m(m)\, a_m(m-k)}{1 - a_m^2(m)}$

i. For m = 3, k = 2,

$$a_2(2) = \frac{a_3(2) - a_3(3)a_3(1)}{1 - a_3^2(3)}$$

$$= \frac{\frac{5}{8} - \frac{1}{3} \times \frac{13}{24}}{1 - \left(\frac{1}{3}\right)^2}$$

$$= \frac{1}{2}$$

ii. For m = 3, k = 1,

$$a_2(1) = \frac{a_3(1) - a_3(3)a_3(2)}{1 - a_3^2(3)}$$

$$= \frac{\frac{13}{24} - \frac{1}{3} \times \frac{1}{2}}{1 - \left(\frac{1}{3}\right)^2}$$

$$= \frac{3}{8}$$

iii. For m = 2, k = 1,

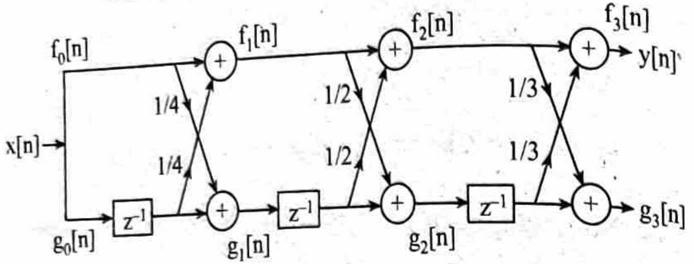
$$a_1(1) = \frac{a_2(1) - a_2(2)a_2(1)}{1 - a_2^2(2)}$$

$$= \frac{\frac{3}{8} - \frac{1}{2} \times \frac{3}{8}}{1 - \left(\frac{1}{2}\right)^2} = \frac{1}{4}$$

So, we have

$k_3 = \frac{1}{3}, k_2 = \frac{1}{2}, k_1 = \frac{1}{4}$

Therefore, the required lattice structure is



Example 4.5:

The system function of a filter is $H(z) = 1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3}$. Draw the direct form and lattice structure implementation of the above filter. **[2078 Bhadra]**

Solution:

Given system function of a filter is

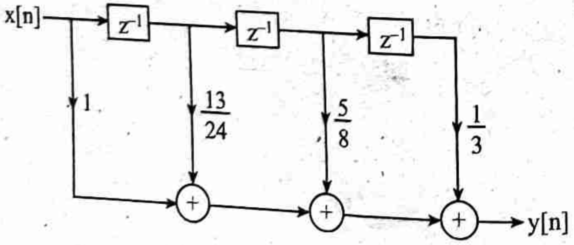
$$H(z) = 1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3} \text{ .... (i)}$$

i. Direct form:

Comparing equation (i) with $H(z) = \sum_{k=0}^{m-1} h[k]z^{-k}$, we have,

$h[0] = 1, h[1] = \frac{13}{24}, h[2] = \frac{5}{8}, h[3] = \frac{1}{3}$

The required direct form structure is



ii. Lattice structure:

Comparing equation (i) with $H(z) = 1 + \sum_{k=1}^{m} a_m(k)z^{-k}$, we have,

Order (m) = 3

and, $a_3(1) = \frac{13}{24}, a_3(2) = \frac{5}{8}, a_3(3) = \frac{1}{3}$

Also, $k_1 = a_1(1) = ?, k_2 = a_2(2) = ?, k_3 = a_3(3) = \frac{1}{3}$

Now, using, $a_{m-1}(k) = \frac{a_m(k) - a_m(m)\, a_m(m-k)}{1 - a_m^2(m)}$

The remaining solution has already been done in the example of lattice.

Example 4.6:

Given a 3-stage lattice filter for all zero polynomial with coefficients $k_1 = \frac{1}{4}$, $k_2 = \frac{1}{2}$, and $k_3 = \frac{1}{3}$, obtain the system function and FIR filter coefficients of this filter.

[2079 Bhadra]

Solution:

Given,

$k_1 = \frac{1}{4}, k_2 = \frac{1}{2}, k_3 = \frac{1}{3}$

Here, order of filter (m) = 3

and,

$a_3(3) = k_3 = \frac{1}{3}$

$a_2(2) = k_2 = \frac{1}{2}$

$a_1(1) = k_1 = \frac{1}{4}$

Now, the general system function of causal FIR system is

$$H(z) = 1 + \sum_{k=1}^{m} a_m(k)z^{-k}$$

$$= 1 + a_3(1)z^{-1} + a_3(2)z^{-2} + a_3(3)z^{-3}$$

Using, $a_m(k) = a_{m-1}(k) + a_m(m)a_{m-1}(m-k)$, we have

i. For m = 2, k = 1,

$$a_2(1) = a_1(1) + a_2(2)a_1(1)$$

$$= \frac{1}{4} + \frac{1}{2} \times \frac{1}{4}$$

$$= \frac{1}{4} + \frac{1}{8}$$

$$= \frac{3}{8}$$

ii. For m = 3, k = 2,

$$a_3(2) = a_2(2) + a_3(3)a_2(1)$$

$$= \frac{1}{2} + \frac{1}{3} \times \frac{3}{8}$$

$$= \frac{1}{2} + \frac{1}{8}$$

$$= \frac{5}{8}$$

iii. For m = 3, k = 1,

$$a_3(1) = a_2(1) + a_3(3)a_2(2)$$

$$= \frac{3}{8} + \frac{1}{3} \times \frac{1}{2}$$

$$= \frac{3}{8} + \frac{1}{6}$$

$$= \frac{13}{24}$$

Hence,

$H(z) = 1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3}$ which is the required system function.

Example 4.7:

Compute the lattice coefficients and draw the lattice structure of the following FIR system.

$H(z) = 1 + 2z^{-1} - 3z^{-2} + 4z^{-3}$. [2072 Kartik, 2069 Chaitra]

Solution:

Given FIR system is

$$H(z) = 1 + 2z^{-1} - 3z^{-2} + 4z^{-3}$$

Comparing with general system function of causal FIR system, $H(z) = 1 + \sum_{k=1}^{m} a_m(k)z^{-k}$, we have

Order of the system (m) = 3

$a_3(1) = 2$, $a_3(2) = -3$, $a_3(3) = 4$

Also,

$k_1 = a_1(1) = ?$, $k_2 = a_2(2) = ?$, $k_3 = a_3(3) = 4$

Now, using $a_{m-1}(k) = \dfrac{a_m(k) - a_m(m)a_m(m-k)}{1 - a_m^2(m)}$

i. For m = 3, k = 2,

$$a_2(2) = \frac{a_3(2) - a_3(3)a_3(1)}{1 - a_3^2(3)}$$

$$= \frac{-3 - 4 \times 2}{1 - (4)^2}$$

$$= \frac{-3 - 8}{1 - 16}$$

$$= \frac{11}{15}$$

ii. For m = 3, k = 1,

$$a_2(1) = \frac{a_3(1) - a_3(3)a_3(2)}{1 - a_3^2(3)}$$

$$= \frac{2 - 4 \times (-3)}{1 - (4)^2}$$

$$= \frac{2 + 12}{1 - 16}$$

$$= \frac{-14}{15}$$

iii. For m = 2, k = 1,

$$a_1(1) = \frac{a_2(1) - a_2(2)a_2(1)}{1 - a_2^2(2)}$$

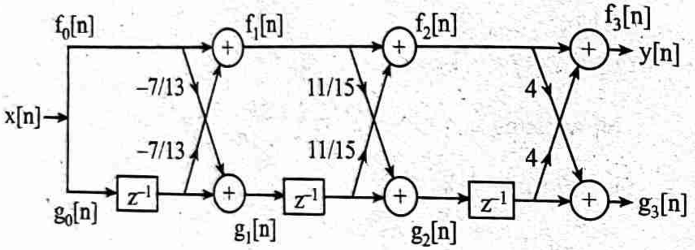
$$= \frac{\frac{-14}{15} - \frac{11}{15} \times \frac{-14}{15}}{1 - \left(\frac{11}{15}\right)^2}$$

$$= \frac{-7}{13}$$

So, we have

$$k_1 = \frac{-7}{13}, k_2 = \frac{11}{15}, k_3 = 4$$

Therefore, the required lattice structure is



4.2 IIR Filter, Structure for IIR Filter (Direct Form I, Direct Form II, Cascade, Lattice, Lattice Ladder)

IIR filter is a type of digital filter that has infinite impulse response, meaning that the filter's response to an impulse input extends indefinitely over time. It is characterized by feedback in their design which allows IIR filter to achieve its desired frequency response with a smaller number of parameters.

IIR filters have lower sidelobes in the stopband than an FIR filter having the same number of parameters.

Structures for IIR Filter

A causal IIR system can be represented by the difference equation:

$$\sum_{k=0}^{N} a_k y(n-k) - \sum_{k=0}^{M} b_k x(n-k) = 0$$

$$\text{or, } a_0 y(n) + \sum_{k=1}^{N} a_k y(n-k) - \sum_{k=0}^{M} b_k x(n-k) = 0$$

For $a_0 = 1$,

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

Taking z-transform, we have

$$Y(z) = -\sum_{k=1}^{N} a_k Y(z) z^{-k} + \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$\text{or, } Y(z) + \sum_{k=1}^{N} a_k Y(z) z^{-k} = \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$\text{or, } Y(z) = \left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] = X(z) \sum_{k=0}^{M} b_k z^{-k}$$

$$\text{or, } \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \quad \text{........ (i)}$$

Direct Form Structure

From equation (i), we have

$$H(z) = \frac{Y(z)}{X(z)}$$

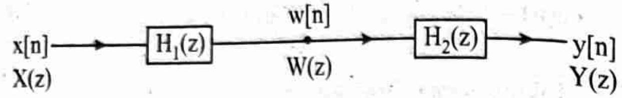
$$= \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=0}^{N} a_k z^{-k}}$$

I. Direct form I:

Suppose $H(z) = H_1(z)H_2(z)$,

where,

$$H_1(z) = \sum_{k=0}^{M} b_k z^{-k},\ H_2 = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}$$



Here, $H_1(z) = \sum_{k=0}^{M} b_k z^{-k}$

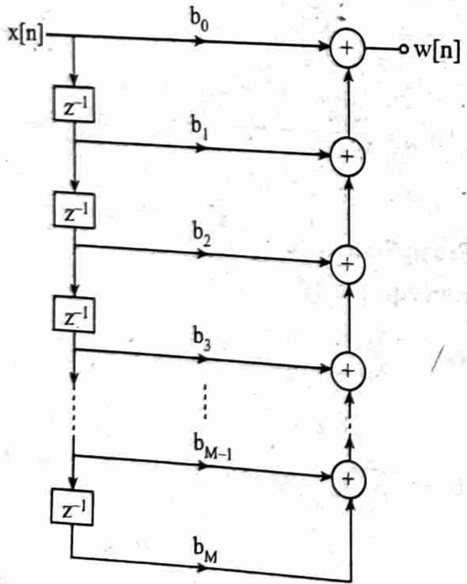
or, $\frac{W(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k}$

or, $W(z) = \sum_{k=0}^{M} b_k X(z) z^{-k}$

$$= b_0 z^0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + b_3 z^{-3} X(z) + \dots + b_M z^{-M} X(z)$$

Taking inverse z-transform,

$w[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + \dots + b_M x[n-M]$



Also, $H_2(z) = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}$

or, $\frac{Y(z)}{W(z)} = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}$

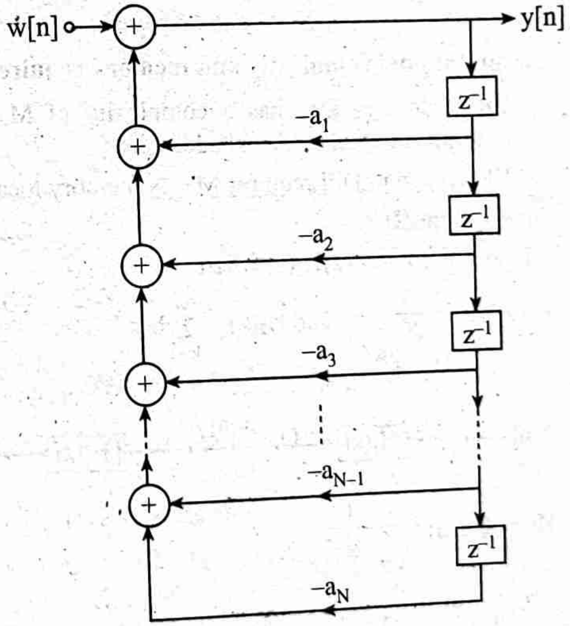
or, $\frac{W(z)}{Y(z)} = 1 + a_1 z^{-1} + a_2 z^{-2} + \dots\dots + a_N z^{-N}$

or, $W(z) = Y(z) + a_1 z^{-1} Y(z) + a_2 Z^{-2} Y(z) + \dots + a_N z^{-N} Y(z)$

or, $Y(z) = W(z) - a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \dots\dots - a_N z^{-N} Y(z)$

Taking inverse z-transform,

$y[n] = w[n] - a_1 y[n-1] - a_2 y[n-2] - \dots - a_N y[n-N]$



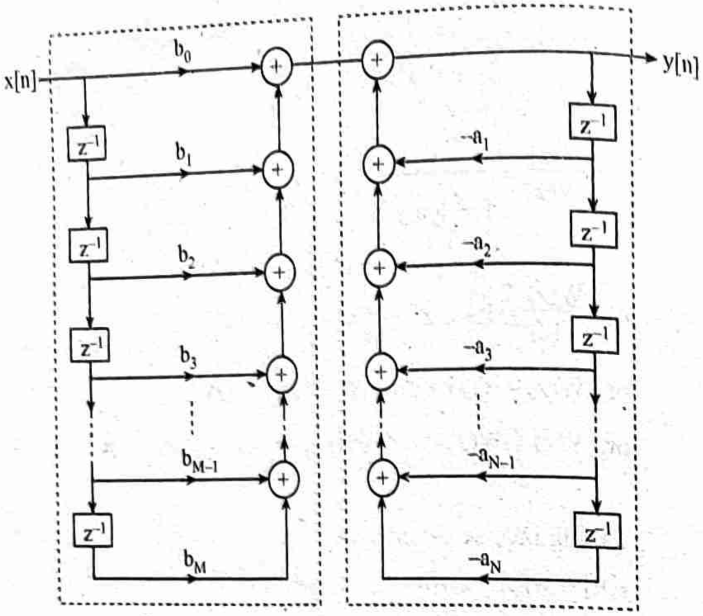


Fig.: Direct form I realization

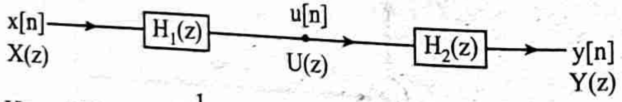
Computational complexity and memory requirement:

- IIR system in DFI has a complexity of M + N + 1 multiplications and M + N additions.
- IIR system in DFI requires M + N memory locations.

II. Direct form II:

Suppose $H(z) = H_1(z)H_2(z)$ where

$$H_1(z) = \frac{1}{1 + \sum\limits_{k=1}^{N} a_k z^{-k}} \text{ and } H_2(z) = \sum_{k=0}^{M} b_k z^{-k}$$



$$\text{Here, } H_1(z) = \frac{1}{1 + \sum\limits_{k=1}^{N} a_k z^{-k}}$$

$$\text{or, } \frac{U(z)}{X(z)} = \frac{1}{1 + \sum\limits_{k=1}^{N} a_k z^{-k}}$$

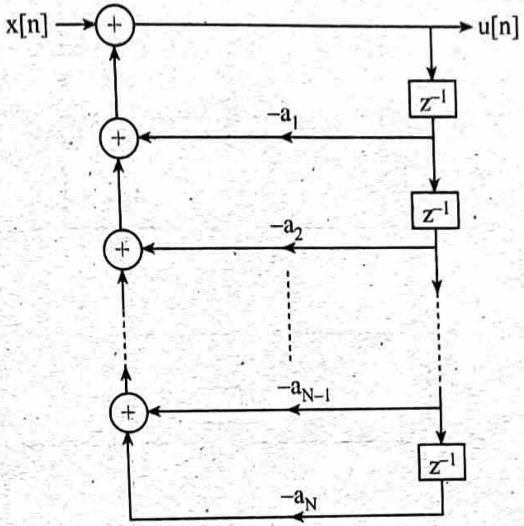
$$\text{or, } \frac{X(z)}{U(z)} = 1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \ldots\ldots + a_N z^{-N}$$

$$\text{or, } X(z) = U(z) + a_1 z^{-1}U(z) + a_2 z^{-2}U(z) + \ldots\ldots + a_N z^{-N}U(z)$$

$$\text{or, } U(z) = X(z) - a_1 z^{-1}U(z) - a_2 z^{-2}U(z) - \ldots\ldots - a_N z^{-N}U(z)$$

Taking inverse z-transform,

$$u[n] = x[n] - a_1 u[n-1] - a_2 u[n-2] - \ldots. - a_N u[n-N]$$



Also,

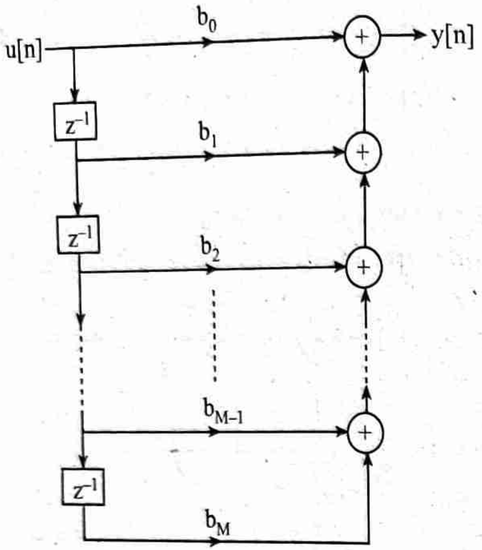
$$H_2(z) = \sum_{k=0}^{M} b_k z^{-k}$$

$$\text{or, } \frac{Y(z)}{U(z)} = b_0 z^0 + b_1 z^{-1} + b_2 z^{-2} + \ldots\ldots + b_M z^{-M}$$

$$\text{or, } Y(z) = b_0 U(z) + b_1 z^{-1}U(z) + b_2 z^{-2}U(z) + \ldots\ldots + b_M z^{-M}U(z)$$

Taking inverse z-transform,

$$y[n] = b_0 u[n] + b_1 u[n-1] + b_2 u[n-2] + \ldots\ldots + b_M u[n-M]$$



So,

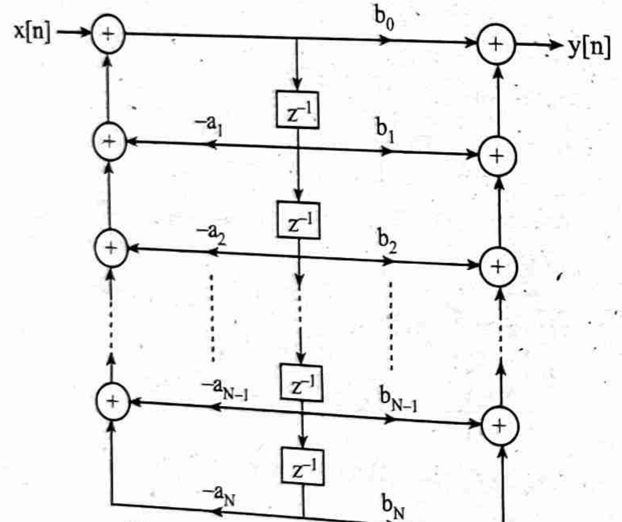


Fig.: Direct form II realization (N = M)

Computational complexity and memory requirement:

- IIR system in DFII has a complexity of M + N + 1 multiplications and M + N additions.
- IIR system in DFII requires largest of [M, N] memory locations.

Example 4.8:

Obtain the Direct Form I and Direct Form II realization of the following IIR digital filter:

$$\mathbf{H(z) = \frac{(0.28z^2 + 0.319z + 0.04)}{(0.5z^3 + 0.3z^2 + 0.17z - 0\ .2)}}$$

Solution:

$$H(z) = \frac{(0.28z^2 + 0.319z + 0.04)}{0.5z^3 + 0.3z^2 + 0.17z - 0\ .2}$$

$$= \frac{z^2(0.28 + 0.319z^{-1} + 0.04z^{-2})}{z^3(0.5 + 0.3z^{-1} + 0.17z^{-2} - 0.2z^{-3})}$$

$$= \frac{z^{-1}(0.28 + 0.319z^{-1} + 0.04z^{-2})}{(0.5 + 0.3z^{-1} + 0.17z^{-2} - 0.2z^{-3})}$$

$$= \frac{0.28z^{-1} + 0.319z^{-2} + 0.04z^{-3}}{0.5(1 + 0.6z^{-1} + 0.34z^{-2} - 0.4z^{-3})}$$

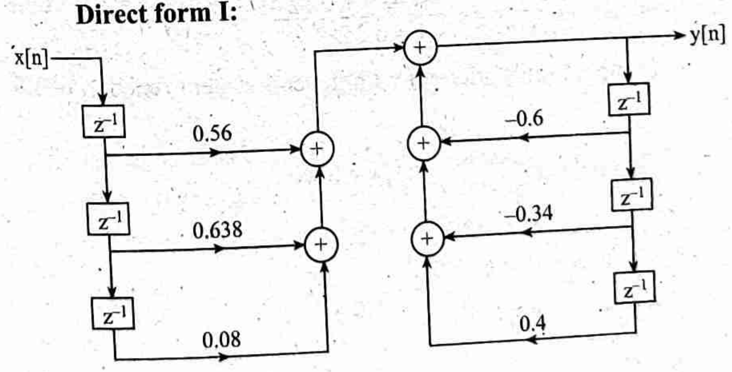
$$= \frac{0.56z^{-1} + 0.638z^{-2} + 0.08z^{-3}}{1 + 0.6z^{-1} + 0.34z^{-2} - 0.4z^{-3}} \text{ .... (i)}$$

Comparing equation (i) with general system function of IIR filter, $H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$, we have

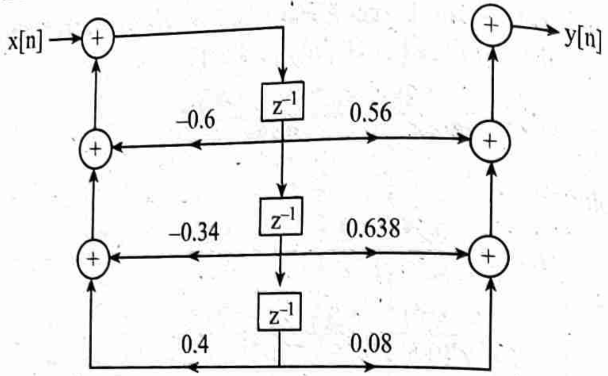
$b_0 = 0, b_1 = 0.56, b_2 = 0.638, b_3 = 0.08$

and, $a_1 = 0.6, a_2 = 0.34, a_3 = -0.4$

Direct form I:



Direct form II:



Example 4.9:

Draw direct form I and direct form II realization of the following system:

$y[n] - 0.25y[n-2] = x[n] + 0.4x[n-1] + 0.5x[n-2]$

[2079 Bhadra]

Solution:

Given, $y[n] - 0.25y[n-2] = x[n] + 0.4x[n-1] + 0.5x[n-2]$

Taking z-transform, we get

$Y(z) - 0.25Y(z)z^{-2} = X(z) + 0.4z^{-1}X(z) + 0.5z^{-2}X(z)$

or, $Y(z)(1 - 0.25z^{-2}) = X(z)\,(1 + 0.4z^{-1} + 0.5z^{-2})$

or, $\frac{Y(z)}{X(z)} = \frac{1+0.4z^{-1}+0.5z^{-2}}{1-0.25z^{-2}}$

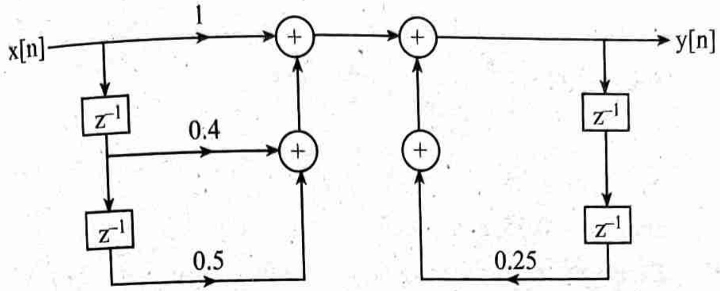
or, $H(z) = \frac{Y(z)}{X(z)} = \frac{1+0.4z^{-1}+0.5z^{-2}}{1-0.25z^{-2}}$ .... (i)

Comparing equation (i) with general system function of IIR filter, $H(z) = \frac{\sum\limits_{k=0}^{M} b_k z^{-k}}{1+\sum\limits_{k=1}^{N} a_k z^{-k}}$, we have

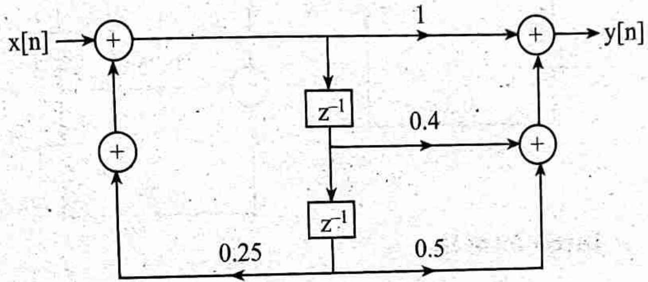
$b_0 = 1, b_1 = 0.4, b_2 = 0.5$

and, $a_1 = 0, a_2 = -0.25$

Direct form I:



Direct form II:



Example 4.10:

Obtain the Direct Form I and Direct Form II realization of the following system:

$y[n] - 0.75y[n-1] - 0.25y[n-2] = x[n] + 0.5x[n-1]$

[2079 Baishakh]

Solution:

Given,

$y[n] - 0.75y[n-1] - 0.25y[n-2] = x[n] + 0.5x[n-1]$

Taking z-transform, we get

$Y(z) - 0.75z^{-1}Y(z) - 0.25z^{-2}Y(z) = X(z) + 0.5z^{-1}X(z)$

or, $Y(z)(1 - 0.75z^{-1} - 0.25z^{-2}) = X(z)(1 + 0.5z^{-1})$

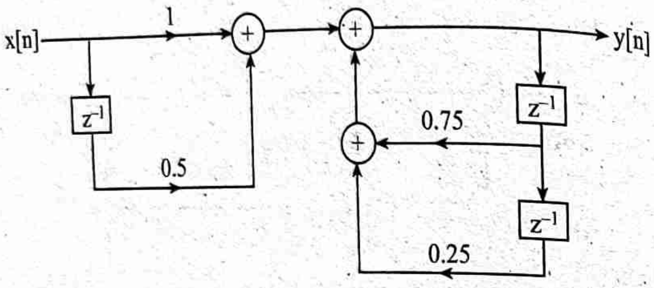
or, $\frac{Y(z)}{X(z)} = \frac{1+0.5z^{-1}}{1-0.75z^{-1}-0.25z^{-2}} = H(z)$ .... (i)

Comparing equation (i) with general system function of IIR filter, $H(z) = \dfrac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$, we have

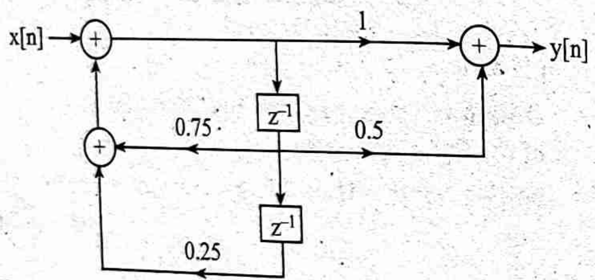
$b_0 = 1, b_1 = 0.5$

and, $a_1 = -0.75, a_2 = -0.25$

Direct form I:



Direct form II:



Example 4.11:

Obtain the Direct Form I and Direct Form II realization of the following system:

$3y[n] + y[n-1] + 2y[n-4] = 2x[n] + x[n-3]$

[2074 Chaitra]

Solution:

Given, $3y[n] + y[n-1] + 2y[n-4] = 2x[n] + x[n-3]$

Taking z-transform, we get

$3Y(z) + z^{-1}Y(z) + 2z^{-4}Y(z) = 2X(z) + z^{-3}X(z)$

or, $Y(z)\,(3 + z^{-1} + 2z^{-4}) = X(z)\,(2 + z^{-3})$

or, $\dfrac{Y(z)}{X(z)} = \dfrac{2 + z^{-3}}{3 + z^{-1} + 2z^{-4}} = H(z)$

or, $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{\frac{2}{3} + \frac{1}{3}z^{-3}}{1 + \frac{1}{3}z^{-1} + \frac{2}{3}z^{-4}}$ .... (i)

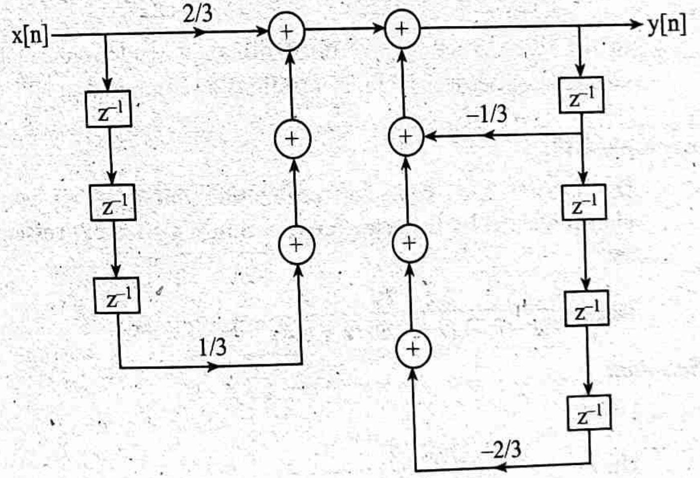
Comparing equation (i) with general system function of IIR filter, $H(z) = \dfrac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$; we have

$b_0 = \frac{2}{3}, b_1 = 0, b_2 = 0, b_3 = \frac{1}{3}$

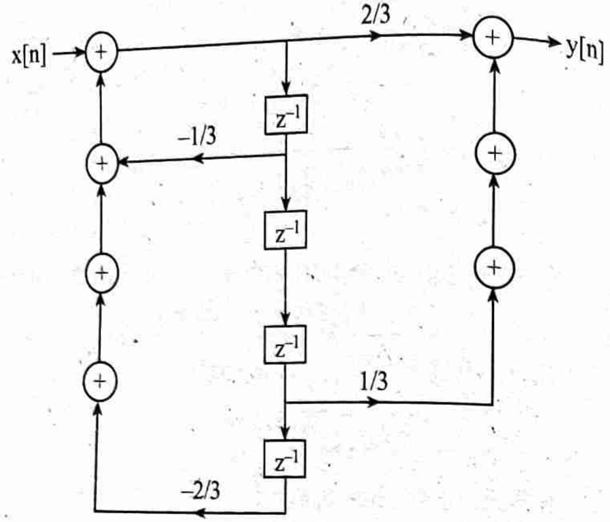
and,

$a_1 = \frac{1}{3}, a_2 = 0, a_3 = 0, a_4 = \frac{2}{3}$

Direct form I:



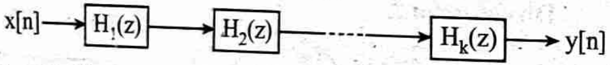
Direct form II:



Cascade Form Structure

In *cascade form*, the total transfer function H(z) is expressed as the product of multiple transfer functions i.e.,

$H(z) = H_1(z).H_2(z).H_3(z) \ldots\ldots\ldots H_k(z)$



In IIR filter structures, we use DFII for cascade realization since it requires less memory locations as compared to DFI.

Example 4.12:

Determine the cascade realization of the system characterized by the transfer function which is expressed as:

$$H(z) = \frac{2(z+2)}{z(z-0.1)(z+0.5)(z+0.4)}$$

Solution:

Given,

$$H(z) = \frac{2(z+2)}{z(z-0.1)(z+0.5)(z+0.4)}$$

$$= \frac{2z(1+2z^{-1})}{zxz^{3}(1-0.1z^{-1})(1+0.5z^{-1})(1+0.4z^{-1})}$$

$$= \frac{2z^{-3}(1+2z^{-1})}{(1-0.1z^{-1})(1+0.5z^{-1})(1+0.4z^{-1})}$$

In cascade form, we can write

$$H(z) = 2z^{-3} \times \frac{(1+2z^{-1})}{(1-0.1z^{-1})} \times \frac{1}{(1+0.5z^{-1})} \times \frac{1}{(1+0.4z^{-1})}$$

or, $H(z) = 2H_1(z)H_2(z)H_3(z)H_4(z)$

Now, for $H_2(z) = \frac{(1+2z^{-1})}{(1-0.1z^{-1})}$, using DFII

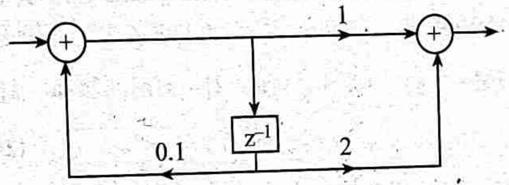
Comparing $H_2(z)$ with general system function of IIR filter,

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \quad \ldots\ldots\ldots \text{(i), we have}$$

$b_0 = 1, b_1 = 2$

and, $a_1 = -0.1$

The required structure is



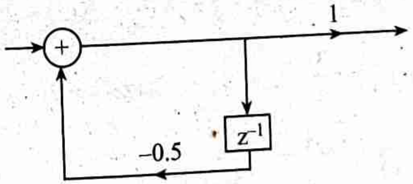
Again, for $H_3(z) = \frac{1}{(1+0.5z^{-1})}$, using DFII

Comparing $H_3(z)$ with equation (i), we have

$b_0 = 1$ and

$a_1 = 0.5$

The required structure is

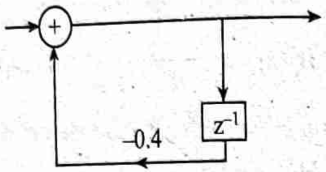


Again, for $H_4(z) = \frac{1}{1 + 0.4z^{-1}}$, using DFII

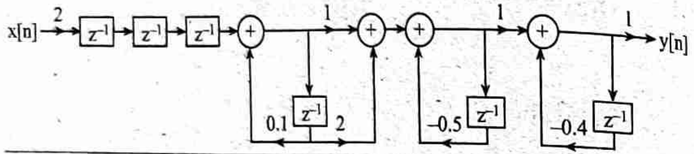
Comparing $H_4(z)$ with equation (i), we have

$b_0 = 1$ and $a_1 = 0.4$

The required structure is



Hence, the overall system structure is



Example 4.13:

Determine the cascade form realization of the following system:

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] - x[n] - 2x[n-1] = 0$$

[2070 Chaitra]

Solution:

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] - x[n] - 2x[n-1] = 0$$

Taking z-transform, we get

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) - X(z) - 2z^{-1}X(z) = 0$$

$$\text{or, } Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = X(z) + 2z^{-1}X(z)$$

$$\text{or, } Y(z)\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right) = X(z)(1 + 2z^{-1})$$

$$\text{or, } \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

$$\text{or, } H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1}}{1 - \left(\frac{1}{2} + \frac{1}{4}\right)z^{-1} + \frac{1}{8}z^{-2}}$$

$$= \frac{1 + 2z^{-1}}{1 - \frac{1}{2}z^{-1} - \frac{1}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

$$= \frac{1 + 2z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right) - \frac{1}{4}z^{-1}\left(1 - \frac{1}{2}z^{-1}\right)}$$

$$= \frac{1 + 2z^{-1}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)} .$$

In cascade form, we can write

$$H(z) = \frac{(1 + 2z^{-1})}{\left(1 - \frac{1}{4}z^{-1}\right)} \times \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)}$$

or, $H(z) = H_1(z).H_2(z)$

We know, the general system function of IIR filter is

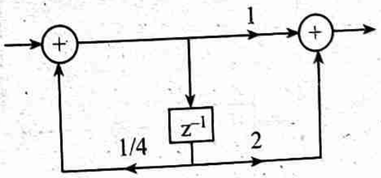
$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \quad \ldots\ldots \text{(i)}$$

For $H_1(z) = \frac{1 + 2z^{-1}}{1 - \frac{1}{4}z^{-1}}$, using DFII

Comparing $H_1(z)$ with equation (i), we have

$b_0 = 1, b_1 = 2$, and $a_1 = -\frac{1}{4}$

The required structure is

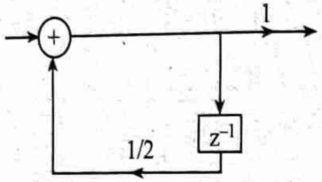


For $H_2(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$, using DFII

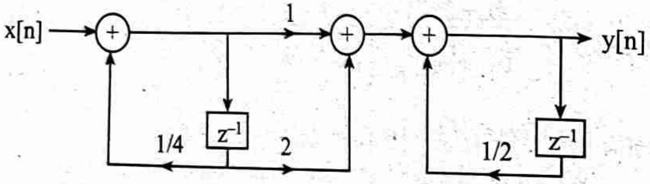
Comparing $H_2(z)$ with equation (i), we have

$b_0 = 1$ and $a_1 = -\frac{1}{2}$

The required structure is



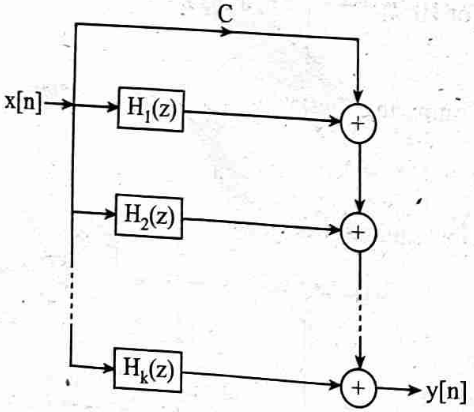
Hence, the overall system structure is



Parallel Form Structure

The parallel form realization for IIR filter is $H(z) = C + H_1(z) + H_2(z) + \ldots\ldots + H_k(z)$. Parallel form realization is generally used for high speed filtering applications.

The structure of parallel form is



Example 4.14:

Obtain the parallel form realization of the following IIR digital filter:

$$\mathbf{H(z) = \frac{3(2z^2 + 5z + 4)}{(2z + 1)(z + 2)}}$$

Solution:

Given, $H(z) = \frac{3(2z^2 + 5z + 4)}{(2z + 1)(z + 2)}$

or, $\frac{H(z)}{z} = \frac{3(2z^2 + 5z + 4)}{z(2z + 1)(z + 2)}$

Using partial fraction, we have

$$F(z) = \frac{H(z)}{z} = \frac{\frac{3}{2}(2z^2 + 5z + 4)}{z\left(z + \frac{1}{2}\right)(z + 2)} = \frac{A}{z} + \frac{B}{\left(z + \frac{1}{2}\right)} + \frac{C}{(z + 2)}$$

$$= \frac{A\left(z + \frac{1}{2}\right)(z + 2) + Bz(z + 2) + Cz\left(z + \frac{1}{2}\right)}{z\left(z + \frac{1}{2}\right)(z + 2)}$$

$$= \frac{A\left(z^2 + \frac{5}{2}z + 1\right) + Bz^2 + 2Bz + Cz^2 + \frac{1}{2}Cz}{z\left(z + \frac{1}{2}\right)(z + 2)}$$

Comparing coefficients, we get

$A + B + C = 3$ .... (i)

and, $\frac{5}{2}A + 2B + \frac{1}{2}C = \frac{15}{2}$ .... (ii)

also, $A = 6$

Using value of A in equations (i) and (ii), we get

$B + C = -3$ .... (a)

and, $2B + \frac{1}{2}C = -\frac{15}{2}$ .... (b)

Solving (a) and (b), we get

$B = -4$ and $C = 1$

Therefore,

$$F(z) = \frac{H(z)}{z} = \frac{6}{2} - \frac{4}{2 + \frac{1}{2}} + \frac{1}{z+2}$$

$$\text{or, } H(z) = 6 - \frac{4z}{z + \frac{1}{2}} + \frac{z}{z+2}$$

$$= 6 - \frac{4}{\left(1 + \frac{1}{2}z^{-1}\right)} + \frac{1}{(1 + 2z^{-1})}$$

We know, the general system function of IIR filter is

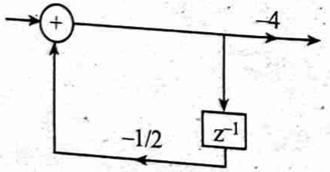
$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \text{ .... (I)}$$

For $H_1(z) = \frac{-4}{1 + \frac{1}{2}z^{-1}}$, using DFII

Comparing $H_1(z)$ with equation (I), we have

$b_0 = -4$ and $a_1 = \frac{1}{2}$

The required structure is,

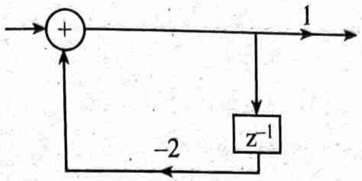


For $H_2(z) = \frac{1}{1 + 2z^{-1}}$, using DFII,

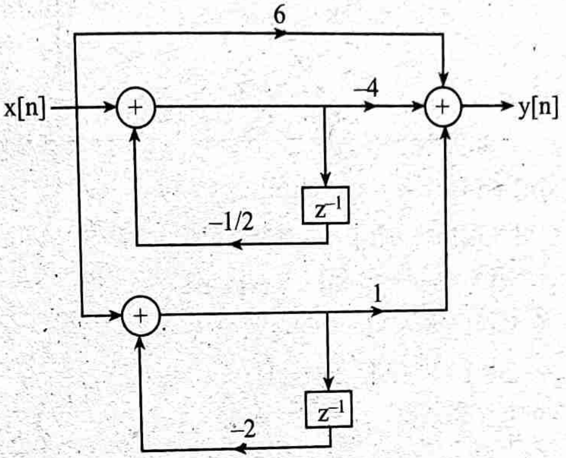
Comparing $H_2(z)$ with equation (I), we have

$a_0 = 1$ and $a_1 = 2$

The required structure is



Hence, the overall system structure is



Lattice Structure of an IIR System

Let us consider an all pole system with system function.

$$H(z) = \frac{1}{1 + \sum_{k=1}^{N} a_N(k) z^{-k}} = \frac{1}{A_N(z)}$$

$$\text{or, } \frac{Y(z)}{X(z)} = \frac{1}{1 + \sum_{k=1}^{N} a_N(k) z^{-k}}$$

$$\text{or, } Y(z) + \sum_{k=1}^{N} a_N(k) z^{-k} Y(z) = X(z)$$

$$\text{or, } Y(z) = X(z) - \sum_{k=1}^{N} a_N(k) z^{-k} Y(z)$$

Taking inverse Fourier transform, we get

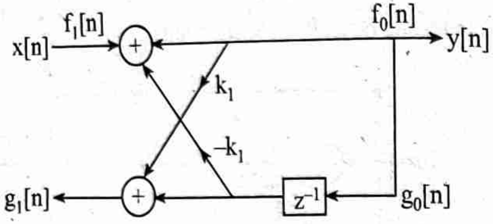
$$y[n] = x[n] - \sum_{k=1}^{N} a_N(k) y[n-k]$$

For N = 1,

$$y[n] = x[n] - \sum_{k=1}^{N} a_1(k)y[n-k]$$

or, $y[n] = x[n] - a_1(1)y[n-1]$

Suppose $a_1(1) = k_1$, then



$x[n] = f_1[n]$

$y[n] = f_0[n] = g_0[n]$

or, $y[n] = f_0[n] = f_1[n] - k_1g_0[n-1]$

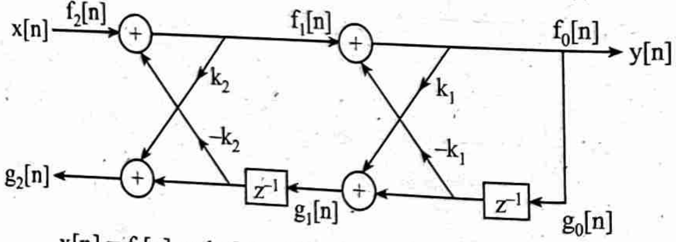
or, $g_1[n] = k_1f_0[n] + g_0[n-1]$

$\Rightarrow$ $g_1[n] = k_1y[n] + y[n-1]$

For N = 2,

$$y[n] = x[n] - \sum_{k=1}^{2} a_2(k)y[n-k]$$

or, $y[n] = x[n] - a_2(1)y[n-1] - a_2(2)y[n-2]$ .... (i)



$x[n] = f_2[n]$ and $y[n] = f_0[n] = g_0[n]$

or, $f_0[n] = f_1[n] - k_1g_0[n-1]$

or, $f_1[n] = f_2[n] - k_2g_1[n-1]$

or, $g_2[n] = k_2f_1[n] + g_1[n-1]$

or, $g_1[n] = k_1f_0[n] + g_0[n-1]$

$\therefore$ $y[n] = f_0[n] = g_0[n] = f_1[n] - k_1g_0[n-1]$

or, $y[n] = f_2[n] - k_2g_1[n-1] - k_1g_0[n-1]$

or, $y[n] = x[n] - k_2g_1[n-1] - k_1y[n-1]$

or, $y[n] = x[n] - k_2\{k_1f_0[n-1] + g_0[n-2]\} - k_1y[n-1]$

or, $y[n] = x[n] - k_1k_2y[n-1] - k_2y[n-2] - k_1y[n-1]$

or, $y[n] = x[n] - k_1(1 + k_2)y[n-1] - k_2y[n-2]$ .... (ii)

Comparing equations (i) and (ii), we get

$a_2(2) = k_2$, $a_2(1) = k_1(1 + k_2)$

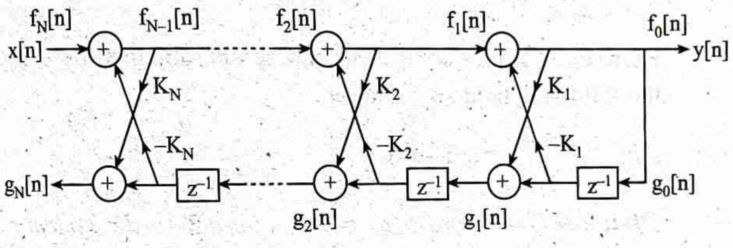
Similarly, for $g_2[n]$,

$g_2[n] = k_2f_1[n] + g_1[n-1]$

On solving, we get

$g_2[n] = k_2x[n] + k_1(1 + k_2)y[n-1] - k_2y[n-2]$

Therefore, N-stage IIR filter is



Lattice Structure Conversion Formula

$k_m = a_m(m)$ where, $1 \le m \le N$

and, $a_{m-1}(k) = \dfrac{a_m(k) - a_m(m)a_m(m-k)}{1 - a_m^2(m)}$

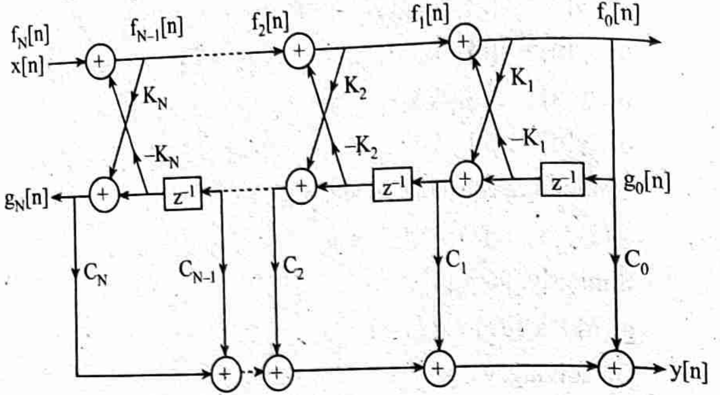
Lattice Ladder Structure of an IIR System

Let us consider an IIR filter with following system function,

$$H(z) = \frac{B_M(k)}{A_N(z)}$$

$$\text{or, } H(z) = \frac{\sum_{k=0}^{M} b_M(k)z^{-k}}{1 + \sum_{k=1}^{N} a_N(k)z^{-k}}$$

The required lattice ladder structure is



The output will be given by

$$y[n] = \sum_{m=0}^{M} c_m g_m(n)$$

Here, c_m is ladder coefficient and can be obtained by using the following recursive relation:

$$c_m = b_m - \sum_{i=m+1}^{M} c_i a_i(i-m)$$

Hint: *For highest order, $c_m = b_m$ i.e. for a 3^rd^ order system $c_3 = b_3$ and remaining c_2, c_1 and c_0 is calculated using the formula.*

Example 4.15:

Convert the following IIR filter into lattice ladder realization:

$$\mathbf{H(z)} = \frac{1+2z^{-1}+2z^{-2}+z^{-3}}{1+\frac{13}{24}z^{-1}+\frac{5}{8}z^{-2}+\frac{1}{3}z^{-3}}$$

Solution:

$$\text{Given, } H(z) = \frac{1+2z^{-1}+2z^{-2}+z^{-3}}{1+\frac{13}{24}z^{-1}+\frac{5}{8}z^{-2}+\frac{1}{3}z^{-3}}$$

Comparing given equation with $H(z) = \dfrac{B_M(z)}{A_N(z)} =$

$$\frac{\sum_{k=0}^{M} b_M(k)z^{-k}}{1+\sum_{k=1}^{N} a_N(k)z^{-k}}$$, we have

$B_M(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3}$; where M = 3

and, $A_N(z) = 1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3}$; where N = 3

So, we can write

$b_3(0) = b_0 = 1$

$b_3(1) = b_1 = 2$

$b_3(2) = b_2 = 2$

$b_3(3) = b_3 = 1$

and,

$$a_3(1) = a_1 = \frac{13}{24}$$

$$a_3(2) = a_2 = \frac{5}{8}$$

$$a_3(3) = a_3 = \frac{1}{3}$$

Now,

$$k_3 = a_3(3) = \frac{1}{3}$$

$k_2 = a_2(2)$

$k_1 = a_1(1)$

Using formula,

$$a_{m-1}(k) = \frac{a_m(k) - a_m(m)a_m(m-k)}{1 - a_m^2(m)}$$

For m = 3, k = 2;

$$a_2(2) = \frac{a_3(2) - a_3(3)a_3(1)}{1 - a_3^2(3)}$$

$$= \frac{\frac{5}{8} - \frac{1}{3} \times \frac{13}{24}}{1 - \left(\frac{1}{3}\right)^2} = \frac{1}{2}$$

For m = 3, k = 1;

$$a_2(1) = \frac{a_3(1) - a_3(3)a_3(2)}{1 - a_3^2(3)}$$

$$= \frac{\frac{13}{24} - \frac{1}{3} \times \frac{5}{8}}{1 - \left(\frac{1}{3}\right)^2}$$

$$= \frac{3}{8}$$

For m = 2, k = 1;

$$a_1(1) = \frac{a_2(1) - a_2(2)a_2(1)}{1 - a_2^2(2)}$$

$$= \frac{\frac{3}{8} - \frac{1}{2} \times \frac{3}{8}}{1 - \left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{4}$$

Therefore,

$k_1 = \frac{1}{4}, k_2 = \frac{1}{2}, k_3 = \frac{1}{3}$

To find ladder coefficients,

$c_m = b_m - \sum_{i=m+1}^{M} c_i a_i(i - m)$ where, $M = 3$

So, $c_3 = b_3 = 1$

Also,

$$c_2 = b_2 - \sum_{i=3}^{3} c_i a_i(i - m)$$

$$= 2 - c_3 a_3(1)$$

$$= 2 - 1 \times \frac{13}{24}$$

$$= \frac{35}{24}$$

$$c_1 = b_1 - \sum_{i=2}^{3} c_i a_i(i - m)$$

$$= 2 - c_2 a_2(1) - c_3 a_3(2)$$

$$= 2 - \frac{35}{24} \times \frac{3}{8} - 1 \times \frac{5}{8}$$

$$= \frac{53}{64}$$

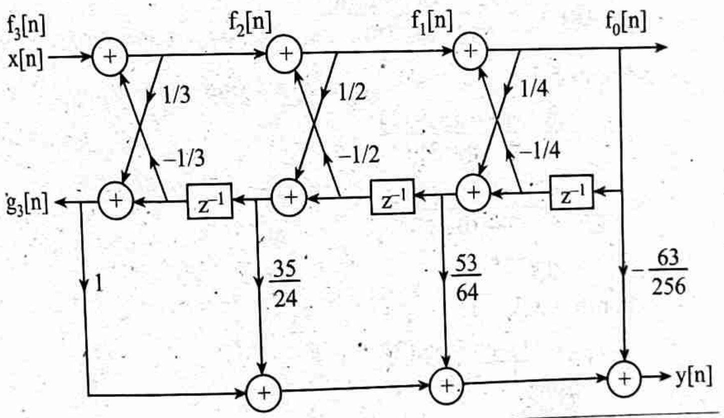
$$c_0 = b_0 - \sum_{i=1}^{3} c_i a_i(i - m)$$

$$= 1 - c_1 a_1(1) - c_2 a_2(2) - c_3 a_3(3)$$

$$= 1 - \frac{53}{64} \times \frac{1}{4} - \frac{35}{24} \times \frac{1}{2} - 1 \times \frac{1}{3}$$

$$= \frac{-69}{256}$$

Hence, the required lattice ladder structure is



Example 4.16:

Draw the lattice structure from the following system function: $H(z) = \frac{1}{1 - 0.2z^{-1} + 0.4z^{-2} + 0.6z^{-3}}$. [2080 Bhadra]

Solution:

Given, $H(z) = \frac{1}{1 - 0.2z^{-1} + 0.4z^{-2} + 0.6z^{-3}}$

Comparing given equation with the equation of an all pole system, $H(z) = \dfrac{1}{1 + \sum\limits_{k=1}^{N} a_N(k)z^{-k}} = \dfrac{1}{A_N(z)}$, we have

$A_N(z) = 1 - 0.2z^{-1} + 0.4z^{-2} + 0.6z^{-3}$; where N = 3

So, we can write

$a_3(1) = a_1 = -0.2$

$a_3(2) = a_2 = 0.4$

$a_3(3) = a_3 = 0.6$

Now,

$k_3 = a_3(3) = 0.6$

$k_2 = a_2(2)$

$k_1 = a_1(1)$

Using formula,

$$a_{m-1}(k) = \frac{a_m(k) - a_m(m)a_m(m-k)}{1 - a_m^2(m)}$$

For m = 3, k = 2;

$$a_2(2) = \frac{a_3(2) - a_3(3)a_3(1)}{1 - a_3^2(3)}$$

$$= \frac{0.4 - 0.6 \times (-0.2)}{1 - (0.6)^2}$$

$$= 0.8125$$

For m = 3, k = 1;

$$a_2(1) = \frac{a_3(1) - a_3(3)a_3(2)}{1 - a_3^2(3)}$$

$$= \frac{-0.2 - 0.6 \times 0.4}{1 - 0.6^2}$$

$$= -0.6875$$

For m = 2, k = 1;

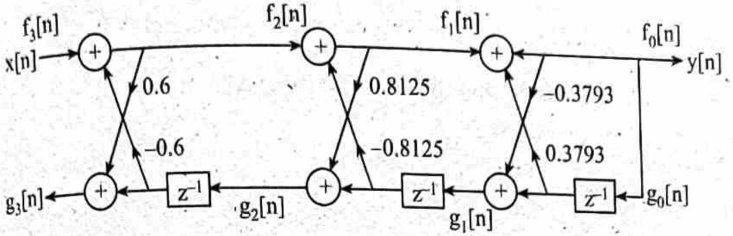
$$a_1(1) = \frac{a_2(1) - a_2(2)a_2(1)}{1 - a_2^2(2)}$$

$$= \frac{-0.6875 - 0.8125 \times (-0.6875)}{1 - (0.8125)^2} = -0.3793$$

Therefore,

$k_1 = -0.3793,\ k_2 = 0.8125,\ k_3 = 0.6$

Hence, the required lattice structure is



Example 4.17:

Draw the lattice structure form of the following system function:

$$H(z) = \frac{1}{3 + \frac{39}{24}z^{-1} + \frac{15}{8}z^{-2} + \frac{3}{9}z^{-3}}$$

And represent $\frac{5}{8}$ and $-\frac{5}{8}$ in sign magnitude, 1's complement and 2's complement format. [2075 Ashwin]

Solution:

$$\text{Given, } H(z) = \frac{1}{3 + \frac{39}{24}z^{-1} + \frac{15}{8}z^{-2} + \frac{3}{9}z^{-3}}$$

$$= \frac{1}{3\left(1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{9}z^{-3}\right)}$$

$$\text{or, } H(z) = \frac{\frac{1}{3}}{1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{9}z^{-3}}$$

Comparing given equation with,

$$H(z) = \frac{B_M(z)}{A_N(z)} = \frac{\sum\limits_{k=0}^{M} b_M(k)z^{-k}}{1 + \sum\limits_{k=1}^{N} a_N(k)z^{-k}},$$

we have,

$B_M(z) = \frac{1}{3}$; where M = 0

and, $A_N(z) = 1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{9}z^{-3}$; where N = 3

We can realize that this is an all pole system.

So, we can write

$$a_3(1) = a_1 = \frac{13}{24}$$

$$a_3(2) = a_2 = \frac{5}{8}$$

$$a_3(3) = a_3 = \frac{1}{9}$$

Now,

$$k_3 = a_3(3) = \frac{1}{9}$$

$k_2 = a_2(2)$

$k_1 = a_1(1)$

Using formula,

$$a_{m-1}(k) = \frac{a_m(k) - a_m(m)a_m(m-k)}{1 - a_m^2(m)}$$

For m = 3, k = 2;

$$a_2(2) = \frac{a_3(2) - a_3(3)a_3(1)}{1 - a_3^2(3)}$$

$$= \frac{\frac{5}{8} - \frac{1}{9} \times \frac{13}{24}}{1 - \left(\frac{1}{9}\right)^2}$$

$$= \frac{183}{320}$$

$$= 0.571875$$

For m = 3, k = 1;

$$a_2(1) = \frac{a_3(1) - a_3(3)a_3(2)}{1 - a_3^2(3)}$$

$$= \frac{\frac{13}{24} - \frac{1}{9} \times \frac{5}{8}}{1 - \left(\frac{1}{9}\right)^2}$$

$$= \frac{153}{320}$$

$$= 0.478125$$

For m = 2, k = 1;

$$a_1(1) = \frac{a_2(1) - a_2(2)a_2(1)}{1 - a_2^2(2)}$$

$$= \frac{0.478125 - 0.571875 \times 0.478125}{1 - (0.571875)^2}$$

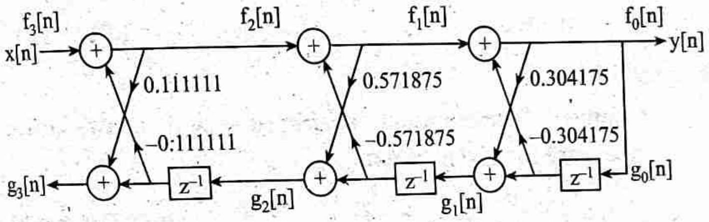
$$= \frac{153}{503}$$

$$= 0.304175$$

Therefore,

$k_1 = 0.304175$, $k_2 = 0.571875$, $k_3 = 0.111111$

Hence, the required lattice structure is



Representation of $\frac{5}{8}$ and $-\frac{5}{8}$ in sign magnitude, 1's complement and 2's complement:

$\left(\frac{5}{8}\right)$

$\frac{5}{8} = 0.625$

In binary,

0.625 = 0.101 ← $\begin{bmatrix}\text{Sign}\\ \text{Magnitude}\end{bmatrix}$ →

$\left(-\frac{5}{8}\right)$

$-\frac{5}{8} = -0.625$

In binary,

–0.625 = 1.101

Conversion of fraction to binary:

$0.625 \times 2 = 1.25 \Rightarrow 1$

$0.25 \times 2 = 0.5 \Rightarrow 0$

$0.5 \times 2 = 1$

1's complement:

In 1's complement, positive numbers are represented as in sign magnitude, but negative numbers are represented by inverting all the bis of the positive number. So,

For $\frac{5}{8}$ or 0.625, For $-\frac{5}{8}$ or -0.625,

$= 0.101$ $= 1.010$

2's complement:

In 2's complement, positive numbers are the same as in sign magnitude, but negative numbers are represented by inverting all the bits of the positive number and then adding 1.

For $\frac{5}{8}$ or 0.625, For $-\frac{5}{8}$ or -0.625,

$= 0.101$ $= 1.011$

Example 4.18:

Compute lattice-ladder coefficients and draw lattice structure for given system.

$$H(z) = \frac{(2 - 0.7z^{-1} + 0.5z^{-2})}{(1 - 0.3z^{-1} + 0.25z^{-2})}$$ **[2080 Baishakh]**

Solution:

Given, $H(z) = \frac{(2 - 0.7z^{-1} + 0.5z^{-2})}{(1 - 0.3z^{-1} + 0.25z^{-2})}$

Comparing given equation with,

$$H(z) = \frac{B_M(z)}{A_N(z)} = \frac{\sum_{k=0}^{M} b_M(k)z^{-k}}{1 + \sum_{k=1}^{N} a_N(k)z^{-k}},$$

we have

$B_M(z) = 2 - 0.7z^{-1} + 0.5z^{-2}$; where M = 2

and, $A_N(z) = 1 - 0.3z^{-1} + 0.25z^{-2}$; where N = 2

So, we can write

$b_2(0) = b_0 = 2$

$b_2(1) = b_1 = -0.7$

$b_2(2) = b_2 = 0.5$

and,

$a_2(1) = -0.3$

$a_2(2) = 0.25$

Now,

$k_2 = a_2(2) = 0.25$

$k_1 = a_1(1)$

Using formula,

$$a_{m-1}(k) = \frac{a_m(k) - a_m(m)a_m(m-k)}{1 - a_m^2(m)}$$

For m = 2, k = 1;

$$a_1(1) = \frac{a_2(1) - a_2(2)a_2(1)}{1 - a_2^2(2)}$$

$$= \frac{-0.3 - 0.25 \times -0.3}{1 - (0.25)^2}$$

$$= -\frac{6}{25}$$

$$= -0.24$$

Therefore,

$k_1 = -0.24$, $k_2 = 0.25$

To find ladder coefficients,

$$c_m = b_m - \sum_{i=m+1}^{M} c_i a_i(i-m);\ M = 2$$

So, $c_2 = b_2 = 0.5$

Also, $c_1 = b_1 - \sum_{i=2}^{2} c_i a_i(i-m)$

$= -0.7 - c_2 a_2(1)$

$= -0.7 - 0.5 \times (-0.3)$

$= -\frac{11}{20}$

$= -0.55$

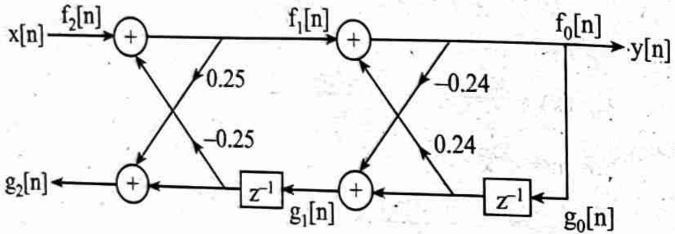
and, $c_0 = b_0 - \sum_{i=1}^{2} c_i a_i(i-m)$

$= 2 - c_1 a_1(1) - c_2 a_2(2)$

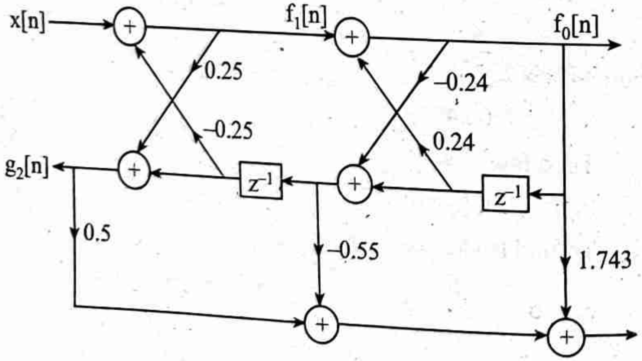
$= 2 - (-0.55) \times (-0.24) - 0.5 \times 0.25$

$= 1.743$

Hence, the required lattice structure is



Note: *The lattice ladder structure if asked:*



Example 4.19:

Compute lattice-ladder coefficients and draw lattice structure for given system, $H(z) = \frac{(1-0.4z^{-1}+0.25z^{-2})}{(1-0.3z^{-1}+0.5z^{-2})}$. Also check the stability of given system. [2079 Baishakh]

Solution:

Given, $H(z) = \frac{(1-0.4z^{-1}+0.25z^{-2})}{(1-0.3z^{-1}+0.5z^{-2})}$

Comparing with $H(z) = \frac{B_M(z)}{A_N(z)} = \frac{\sum_{k=0}^{M} b_M(k)z^{-k}}{1 + \sum_{k=1}^{N} a_N(k)z^{-k}}$, we have

$B_M(z) = 1 - 0.4z^{-1} + 0.25z^{-2}$; where M = 2

and, $A_N(z) = 1 - 0.3z^{-1} + 0.5z^{-2}$; where N = 2

So, we can write

$b_2(0) = b_0 = 1$

$b_2(1) = b_1 = -0.4$

$b_2(2) = b_2 = 0.25$

and,

$a_2(1) = -0.3$

$a_2(2) = 0.5$

Now,

$k_2 = a_2(2) = 0.5$

$k_1 = a_1(1)$

Using formula,

$a_{m-1}(k) = \frac{a_m(k) - a_m(m)a_m(m-k)}{1 - a_m^2(m)}$

For m = 2, k = 1;

$a_1(1) = \frac{a_2(1) - a_2(2)a_2(1)}{1 - a_2^2(2)}$

$= \frac{-0.3 - 0.5 \times (-0.3)}{1 - (0.5)^2}$

$$= -\frac{1}{5} = -0.2$$

Therefore,

$k_1 = -0.2, k_2 = 0.5$

To find ladder coefficients,

$$c_m = b_m - \sum_{i=m+1}^{M} c_i a_i(i-m); \text{ where } M = 2$$

So, $c_2 = b_2 = 0.25$

Also, $c_1 = b_1 - \sum_{i=2}^{2} c_i a_i(i-m)$

$= -0.4 - c_2 a_2(1)$

$= -0.4 - 0.25 \times -0.3$

$= -0.325$

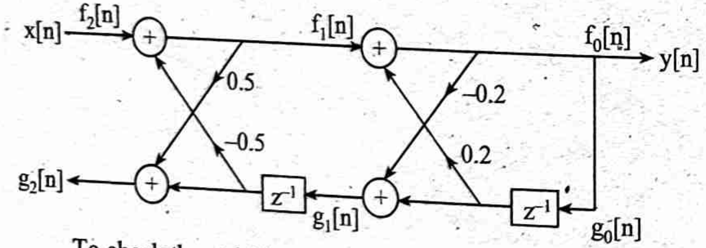
and, $c_0 = b_0 - \sum_{i=2}^{2} c_i a_i(i-m)$

$= 1 - c_1 a_1(1) - c_2 a_2(2)$

$= 1 - (-0.325) \times (-0.2) - 0.25 \times 0.5$

$= 0.81$

Hence, the required lattice structure is



To check the stability of the system:

Given, $H(z) = \frac{1 - 0.4z^{-1} + 0.25z^{-2}}{1 - 0.3z^{-1} + 0.5z^{-2}}$

or, $H(z) = \frac{z^2 - 0.4z + 0.25}{z^2 - 0.3z + 0.5}$

Here,

Poles: $z^2 - 0.3z + 0.5 = 0$

$\Rightarrow z = 0.15 + j0.691, 0.15 - j0.691$

$z = 0.15, 0.4775$

Zeros: $z^2 - 0.4z + 0.25 = 0$

$\Rightarrow z = 0.2 + j0.458, 0.2 - j0.458$

For a discrete time system to be stable, all poles must lie inside the unit circle in the complex plane (i.e. the magnitude of all poles must be less than 1).

Given pole of the system is

$z = 0.15 \pm j0.691$

Magnitude $= \sqrt{(0.15)^2 + (0.691)^2} = 0.707$

Since, $|z| = 0.707 < 1$, the poles lie inside the unit circle and hence, the system is stable.

Example 4.20:

Compute lattice and ladder coefficients and draw lattice ladder structure for given IIR system:

$$\mathbf{H(z) = \frac{(0.5 - 2z^{-1} + 3z^{-2})}{(1 - 0.5z^{-1} - 0.7z^{-2} + 0.3z^{-3})}}$$ **[2076 Chaitra]**

Solution:

Given, $H(z) = \frac{(0.5 - 2z^{-1} + 3z^{-2})}{(1 - 0.5z^{-1} - 0.7z^{-2} + 0.3z^{-3})}$

Comparing given equation with H(z)

$$= \frac{B_M(z)}{A_N(2)} = \frac{\sum_{k=0}^{M} b_M(k)z^{-k}}{1 + \sum_{k=1}^{N} a_N(k)z^{-k}},$$

we have

$B_M(z) = 0.5 - 2z^{-1} + 3z^{-2}$; where $M = 2$

and, $A_N(z) = 1 - 0.5z^{-1} - 0.7z^{-2} + 0.3z^{-3}$; where $N = 3$

So, we can write

$b_2(0) = b_0 = 0.5$

$b_2(1) = b_1 = -2$

$b_2(2) = b_2 = 3$

and,

$a_3(1) = -0.5$

$a_3(2) = -0.7$

$a_3(3) = 0.3$

Now,

$k_3 = a_3(3) = 0.3$

$k_2 = a_2(2)$

$k_1 = a_1(1)$

Using formula,

$$a_{m-1}(k) = \frac{a_m(k) - a_m(m)a_m(m-k)}{1 - a_m^2(m)}$$

For m = 3, k = 2;

$$a_2(2) = \frac{a_3(2) - a_3(3)a_3(1)}{1 - a_3^2(3)}$$

$$= \frac{-0.7 - 0.3 \times (-0.5)}{1 - (0.3)^2}$$

$$= -0.6044$$

For m = 3, k = 1;

$$a_2(1) = \frac{a_3(1) - a_3(3)a_3(2)}{1 - a_3^2(3)}$$

$$= \frac{-0.5 - 0.3 \times (-0.7)}{1 - (0.3)^2}$$

$$= -0.3187$$

For m = 2, k = 1;

$$a_1(1) = \frac{a_2(1) - a_2(2)a_2(1)}{1 - a_2^2(2)}$$

$$= \frac{-0.3187 - (-0.6044) \times (-0.3187)}{1 - (-0.6044)^2}$$

$$= -0.8056$$

Therefore,

$k_1 = -0.8056$, $k_2 = -0.6044$, $k_3 = 0.3$

To find ladder coefficients,

$$c_m = b_m - \sum_{i=m+1}^{M} c_i a_i(i-m); \text{ where } M = 2$$

So, $c_2 = b_2 = 3$

Also, $c_1 = b_1 - \sum_{i=2}^{2} c_i a_i(i-m)$

$$= -2 - c_2 a_2(1)$$

$$= -2 - 3 \times (-0.3187)$$

$$= -1.0439$$

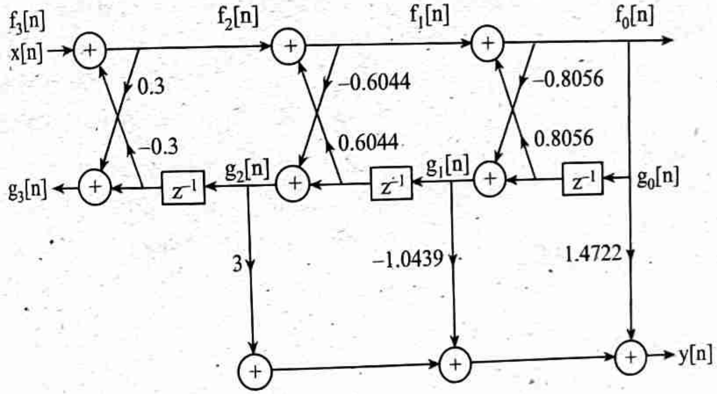
and, $c_0 = b_0 - \sum_{i=1}^{2} c_i a_i(i-m)$

$$= 0.5 - c_1 a_1(1) - c_2 a_2(2)$$

$$= 0.5 - (-1.0439) \times (-0.8056) - 3 \times (-0.6044)$$

$$= 1.4722$$

Hence, the required lattice ladder structure is



Differences between FIR System and IIR System

[2080 Baishakh]

FIR system	IIR system
i. FIR system are non-recursive system having finite impulse response.	i. IIR system are recursive system having infinite impulse response.
ii. Feedback system is absent.	ii. Feedback system is present.

FIR system	IIR system
iii. It is always stable due to absence of poles.	iii. It can be unstable due to presence of poles.
iv. It typically requires more memory.	iv. It requires less memory due to feedback path.
v. Generally higher computational complexity.	v. Generally lower computational complexity.
vi. The transfer function contains only zeros.	vi. The transfer function contains both poles and zeros.
vii. Designed using windowing method.	vii. Designed using various methods such as Butterworth Chebyshev, etc.

Chapter – 5

FIR FILTER DESIGN

5.1 Filter Design by Window Method, Commonly Used Windows (Rectangular Window, Hanning Window, Hamming Window)

There are several methods for designing FIR or IIR filters. The type of filter (either FIR or IIR) to be chosen depends on the nature of the problem and on the specifications of the desired frequency response.

The desired filter characteristics are specified in the frequency domain in terms of the desired magnitude and phase response of the filter. Designing the filter involves determining the coefficients of causal FIR and IIR filter that closely approximate the desired frequency response.

FIR filters are employed where there is a requirement of linear phase in passband. However, IIR filters have lower side lobes in the stopband than an FIR filter having the same number of parameter. The choice of filter depends on the need.

Let us consider a digital filter that is to be designed have the frequency response specification $H_d(\omega)$. This is also called desired frequency response. The corresponding unit sample response is $h_d[n]$.

We know,

$$H_d(\omega) = \sum_{n=0}^{\infty} h_d[n]e^{-j\omega n}$$

where,

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}d\omega$$

Thus, the desired unit sample response $h_d[n]$ can be obtained from the desired frequency response $H_d(\omega)$.

In general, the unit sample response $h_d[n]$ is infinite in duration. Since, we are designing a finite impulse response filter, the length of $h_d[n]$ must be made finite. For this, $h_d[n]$ is truncated to a series from $n = 0$ to $n = M - 1$, to yield an FIR filter of length M. Truncation of $h_d[n]$ to a length $M - 1$ is equivalent to multiplying $h_d[n]$ by a rectangular window.

The rectangular window is represented as $w_R[n]$ and is expressed as:

$$w_R[n] = \begin{cases} 1; & \text{for } n = 0, 1, \ldots\ldots, M-1 \\ 0; & \text{otherwise.} \end{cases}$$

Thus, the unit sample response of the FIR filter becomes

$$h[n] = h_d[n]\, w_R[n]$$

$$\Rightarrow h[n] = \begin{cases} h_d[n]; & \text{for } n = 0, 1, \ldots\ldots, M-1 \\ 0; & \text{otherwise} \end{cases}$$

The Fourier transform of the rectangular window is

$$W_R(\omega) = \sum_{n=0}^{M-1} w_R[n]e^{-j\omega n}$$

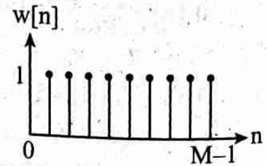
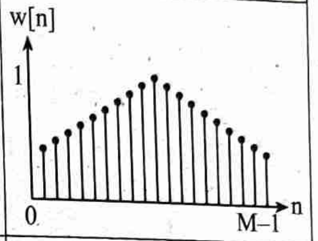
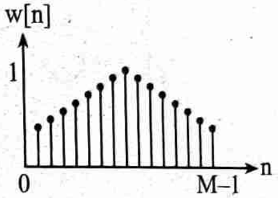
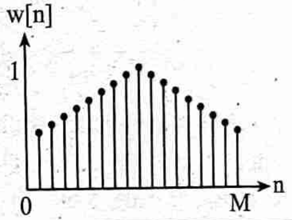
$$= \sum_{n=0}^{M-1} e^{-j\omega n}$$

$$= \frac{1-e^{-j\omega M}}{1-e^{-j\omega}} = e^{-j\omega(M-1)/2}\,\frac{\sin(\omega M/2)}{\sin(\omega/2)}$$

The magnitude response of rectangular window is

$$|W_R(\omega)| = \left|\frac{\sin(\omega M/2)}{\sin(\omega/2)}\right|$$

Window Functions for FIR Filter Design

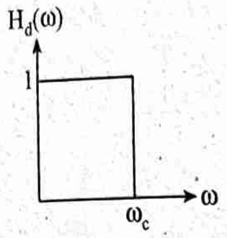
Name of window	Time domain sequence $w[n], 0 \le n \le M-1$	Shape of window function
i. Rectangular	1	
ii. Hamming	$0.54-0.46\cos\left(\frac{2\pi n}{M-1}\right)$	
iii. Hanning	$\frac{1}{2}\left[1-\cos\left(\frac{2\pi n}{M-1}\right)\right]$	
iv. Kaiser	$\frac{I_0\left\{\beta\left[1-\left(\frac{n-\alpha}{\alpha}\right)^2\right]^{1/2}\right\}}{I_0(\beta)}$ $I_0(\beta)$ is modified Bessel function.	

Relation of Windows with Sideband (SB) Attenuation and Value of M

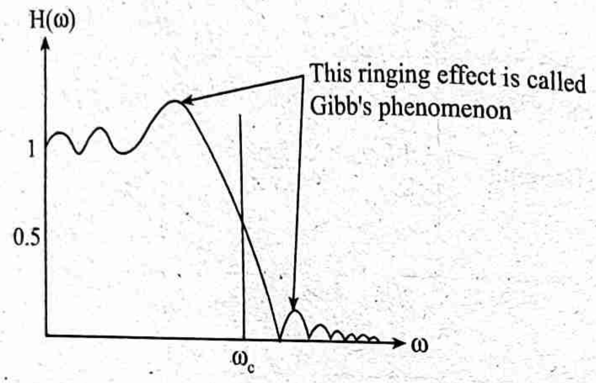
Name of window	SB attenuation	Value of M
i. Rectangular	~21dB	$M = 0.9\frac{2\pi}{\Delta\omega}$
ii. Hamming	~53dB	$M = 3.3\frac{2\pi}{\Delta\omega}$
iii. Hanning	~44dB	$M = 3.1\frac{2\pi}{\Delta\omega}$

Gibbs Phenomenon in FIR Filter Design

Let us consider a low pass filter having desired frequency response $H_d(\omega)$ as shown in the figure below.



The obtained frequency response of FIR filter is $H(\omega)$.



In the figure, oscillations or ringing takes place near band edge (ω_c) of the filter. These oscillations or ringing is generated because of side lobes in the frequency response of the window function. This oscillatory behaviour (i.e. ringing effect) is known as *Gibbs phenomenon.*

Thus, ringing effect takes place because of side lobes in w[n]. These side lobes are generated because of abrupt discontinuity of the window function. In case of rectangular window, the side lobes are larger in size because the discontinuity is abrupt (sudden and unexpected). Therefore, the ringing effect is maximum in rectangular window.

Because of this, different window functions are developed which consists of taper and decays gradually towards zero. This reduces side lobes and hence ringing effect in $H(\omega)$.

The Gibb's phenomenon can be reduced under following conditions:

i. The central lobe of window must not be wide.

ii. The side lobe levels must be as small as possible.

iii. The central lobe must contain maximum energy.

iv. Ripples should be minimum.

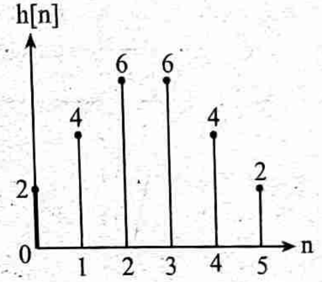
So, Gibb's phenomenon can be reduced to some extent by selection of window.

Symmetric and Anti-Symmetric FIR Filters

FIR filter is defined symmetric and anti-symmetric in terms of h[n].

Symmetric: If it satisfies

$h[n] = h[M - 1 - n]$; where n = 0, 1, ..., M–1

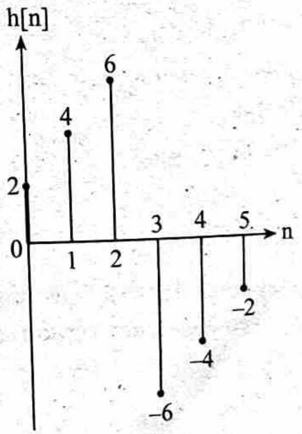


Here, M = 6

$h[0] = 2, h[1] = 4, h[2] = 6, h[3] = 6, h[4] = 4, h[5] = 2$

Anti-symmetric: If it satisfies

$h[n] = -h[M - 1 - n]$; where n = 0, 1, ..., M – 1

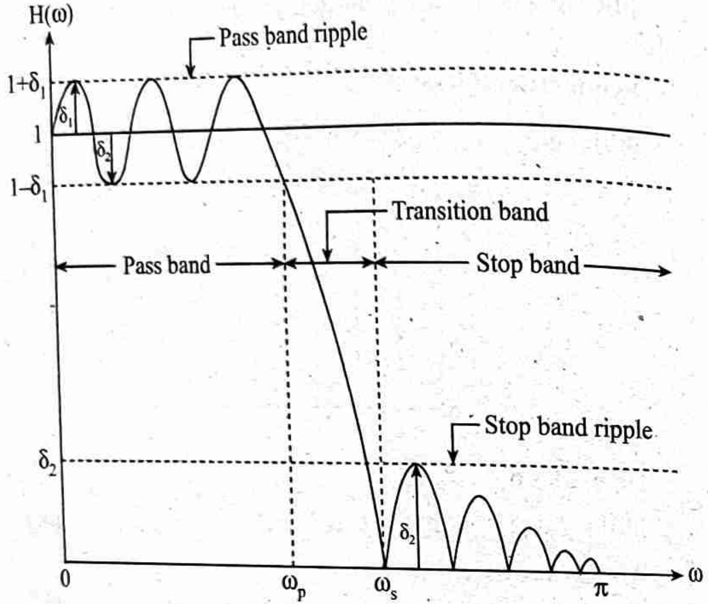


Here, M = 6

$h[0] = 2, h[1] = 4, h[2] = 6, h[3] = -6, h[4] = -4, h[5] = -2$

***Note:** Linear phase filter satisfies the condition for symmetry or anti-symmetry. Hence, for a linear phase filter, $h[n] = \pm h[m-1-n]$*

Magnitude Characteristics of Practical Filters



$$1-\delta_1 \le |H(w)| \le 1+\delta_1 \text{ for } 0 \le w \le w_p$$

$$0 \le |H(w)| \le \delta_2 \text{ for } w_s \le w \le \pi.$$

The desired frequency response for a symmetric low pass linear phase FIR filter is

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & \text{for } |\omega| \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Example 5.1:

Design the symmetric FIR low pass filter (LPF) for which the desired frequency response is expressed as

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau}; & |\omega| \le \omega_c \\ 0; & \text{elsewhere} \end{cases}$$

The length of the filter should be 7 and $\omega_c = 1$ rad/sample. Make use of the Hanning window.

[2080 Bhadra]

Solution:

Given,

Length of the filter, M = 7

Cutoff frequency, $\omega_c = 1$ rad/sample

Desired frequency response,

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau}; & |\omega| \le \omega_c \\ 0; & \text{elsewhere} \end{cases}$$

We know,

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n} d\omega$$

$$\text{So, } h_d[n] = \frac{1}{2\pi}\int_{-1}^{1} e^{-j\omega\tau} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1} e^{j\omega(n-\tau)}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{-1}^{1}$$

$$= \frac{1}{2\pi}\left[\frac{e^{j(n-\tau)}}{j(n-\tau)} - \frac{e^{-j(n-\tau)}}{j(n-\tau)}\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{j(n-\tau)} - e^{-j(n-\tau)}}{j(n-\tau)}\right]$$

$$= \frac{1}{\pi(n-\tau)}\left[\frac{e^{j(n-\tau)} - e^{-j(n-\tau)}}{2j}\right]$$

$$= \frac{1}{\pi(n-\tau)} \sin(n-\tau)$$

$$\therefore\ h_d[n] = \frac{\sin(n-\tau)}{\pi(n-\tau)} \text{ for } n \ne \tau$$

When $n = \tau$, $h_d[n] = \frac{1}{2\pi}\int_{-1}^{1} d\omega$

$$= \frac{1}{2\pi}[\omega]_{-1}^{1}$$

$$= \frac{1}{2\pi}[1 - (-1)]$$

$$= \frac{2}{2\pi}$$

$$= \frac{1}{\pi}$$

Therefore,

$$h_d[n] = \begin{cases} \frac{\sin(n-\tau)}{\pi(n-\tau)}; & \text{for } n \neq \tau \\ \frac{1}{\pi}; & \text{for } n = \tau \end{cases}$$

To find the value of τ:

For symmetric filter, we know

$h[n] = h[M - 1 - n]$

Also, $h[n] = h_d[n]\, w[n]$

So, $h_d[n]\, w[n] = h_d[M - 1 - n]\, w[n]$

$\Rightarrow h_d[n] = h_d[M - 1 - n]$

or, $\frac{\sin(n-\tau)}{\pi(n-\tau)} = \frac{\sin(M-1-n-\tau)}{\pi(M-1-n-\tau)}$

This above condition satisfies only if

$-(n - \tau) = M - 1 - n - \tau$

or, $-n + \tau = M - 1 - n - \tau$

or, $2\tau = M - 1$

or, $\tau = \frac{M-1}{2}$

Since $m = 7$, $\tau = 3$

$$\text{Hence, } h_d[n] = \begin{cases} \frac{\sin(n-3)}{\pi(n-3)}; & \text{for } n \neq 3 \\ \frac{1}{\pi}; & \text{for } n = 3 \end{cases}$$

For Hanning window:

n	$h_d[n] = \frac{\sin(n-3)}{\pi(n-3)}$; $n \neq 3$ $= \frac{1}{\pi}$; $n = 3$	$w[n] = \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{M-1}\right)\right]$ $0 \leq n \leq 6$	$h[n] = h_d[n]w[n]$
0	0.01497	0	0
1	0.14472	0.25	0.03618
2	0.26785	0.75	0.20088
3	0.3183	1	0.3183
4	0.26785	0.75	0.20088
5	0.14472	0.25	0.03618
6	0.01497	0	0

Hence,

$h[0] = 0$, $h[1] = 0.03618$, $h[2] = 0.20088$, $h[3] = 0.3183$,

$h[4] = 0.20088$, $h[5] = 0.03618$, $h[6] = 0$

$$\therefore\ h[n] = \{\underset{\uparrow}{0}, 0.03618, 0.20088, 0.3183, 0.20088, 0.03618, 0\}$$

For rectangular window:

n	$h_d[n] = \frac{\sin(n-3)}{\pi(n-3)}$; $n \neq 3$ $= \frac{1}{\pi}$; $n = 3$	$w[n] = 1$; $0 \leq n \leq 6$ $= 0$; otherwise	$h[n] = h_d[n]w[n]$
0	0.01497	1	0.01497
1	0.14472	1	0.14472
2	0.26785	1	0.26785
3	0.3183	1	0.3183

n	$h_d[n] = \frac{\sin(n-3)}{\pi(n-3)}$; $n \neq 3$ $= \frac{1}{\pi}$; $n = 3$	$w[n] = 1; 0 \le n \le 6$ $= 0$; otherwise	$h[n] = h_d[n]w[n]$
4	0.26785	1	0.26785
5	0.14472	1	0.14472
6	0.01497	1	0.01497

Hence,

h[0] = 0.01497, h[1] = 0.14472, h[2] = 0.26785, h[3] = 0.3183, h[4] = 0.26785, h[5] = 0.14472, h[6] = 0.01497

∴ h[n]

$$= \{\underset{\uparrow}{0.01497}, 0.14472, 0.26785, 0.3183, 0.26785, 0.14472, 0.01497\}$$

For Hamming window:

n	$h_d[n] = \frac{\sin(n-3)}{\pi(n-3)}$; $n \neq 3$ $= \frac{1}{\pi}$; $n = 3$	$w[n] = 0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right)$ $0 \le n \le 6$	$h[n] = h_d[n]w[n]$
0	0.01497	0.08	0.001198
1	0.14472	0.31	0.044863
2	0.26785	0.77	0.206244
3	0.3183	1	0.3183
4	0.26785	0.77	0.206244
5	0.14472	0.31	0.044863
6	0.01497	0.08	0.001198

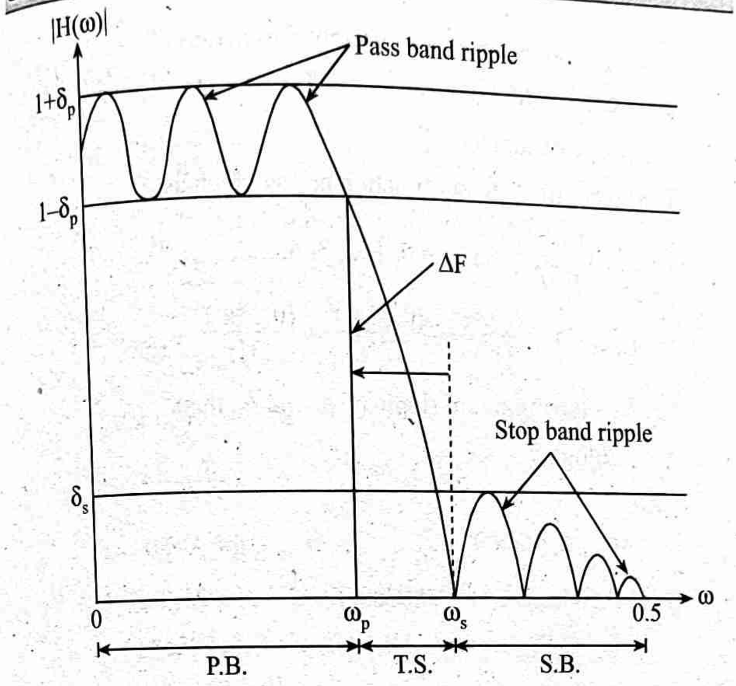
Hence,

h[0] = 0.001198, h[1] = 0.044863, h[2] = 0.206244, h[3] = 0.3183, h[4] = 0.206244, h[5] = 0.044863, h[6] = 0.001198

∴ h[n]

$$= \{\underset{\uparrow}{0.001198}, 0.044863, 0.206244, 0.318, 0.206244, 0.044863, 0.001198\}$$

5.2 Filter Design by Kaiser Window



P.B. = Pass band

T.B. = Transition band

S.B. = Stop band

δ_p = Peak passband deviation

δ_s = Stopband deviation

ω_p = Passband edge frequency

ω_s = Stopband edge frequency

$$\Delta\omega = \omega_s - \omega_p,\ \omega_c = \frac{(\omega_s + \omega_p)}{2}$$

$$\Delta F = \text{Transition width} = \frac{\Delta\omega}{2\pi} = \frac{\omega_s - \omega_p}{2\pi}$$

Kaiser window is given by

$$w[n] = \frac{I_0\left\{\beta\left[1 - \left(\frac{n-\alpha}{\alpha}\right)^2\right]^{1/2}\right\}}{I_0(\beta)} \text{ for } 0 \le n \le M$$

Here, $\alpha = \frac{M}{2}$

Kaiser window has two important parameters:

i. Length = M + 1

ii. Shape parameter = β

I_0 is a modified Bessel function and is given as

$$I_0(x) = \frac{(0.25x^2)^n}{(n!)^2} \text{ for } n = 0, 1, 2, 3, 4, \ldots\ldots$$

$$\Rightarrow I_0(x) = 1 + \frac{0.25x^2}{(1!)^2} + \frac{(0.25x^2)^2}{(2!)^2} + \frac{(0.25x^2)^3}{(3!)^2} + \ldots\ldots$$

Let δ be the minimum ripple of δ_p and δ_s, then

$$A = -20\log_{10}\delta$$

Also,

$$\beta = \begin{cases} 0.1102(A-8.7) & \text{for } A>50 \\ 0.5842(A-21)^{0.4}+0.07886(A-21) & \text{for } 21\le A\le 50 \\ 0 & \text{for } A < 21 \end{cases}$$

And, $M = \frac{A-8}{22.85\Delta\omega}$

The order of filter is M + 1.

Example 5.2:

Design a linear phase FIR filter using Kaiser Window to meet the following specifications:

$$\mathbf{0.99 \le |H(e^{j\omega})| \le 1.01, \quad \text{for } 0 \le \omega \le 0.19\pi}$$

$$\mathbf{|H(e^{j\omega})| \le 0.01, \quad \text{for } 0.21\pi \le \omega \le \pi}$$

[2075 Ashwin, 2072 Kartik]

Solution:

The above given specifications can be rewritten as:

$$1 - \delta_1 \le |H(\omega)| \le 1 + \delta_1 \quad \text{for } 0 \le \omega \le \omega_p$$

$$0 \le |H(\omega)| \le \delta_2 \quad \text{for } \omega_s \le \omega \le \pi \quad \ldots\ldots\ (I)$$

Comparing given specification with (I), we have

$\delta_1 = 0.01$, $\omega_p = 0.19\pi$, $\delta_2 = 0.01$, $\omega_s = 0.21\pi$

We know,

$\delta = \text{minimum}(\delta_1, \delta_2) = 0.01$

$A = -20\log_{10}(\delta) = 40$ dB

$$M = \frac{A-8}{2.285\Delta\omega}$$

where, $\Delta\omega = \omega_s - \omega_p = 0.21\pi - 0.19\pi = 0.02\pi$

$$= \frac{40-8}{2.285 \times 0.02\pi}$$

$$= 222.88657 \simeq 223$$

Order of filter = M + 1 = 224

$$\omega_c = \frac{\omega_s + \omega_p}{2} = 0.2\pi$$

Here, for $21 \le A \le 50$, we have

$$\beta = 0.5842(A-21)^{0.4} + 0.07886\,(A-21)$$

$$= 0.5842(40-21)^{0.4} + 0.07886\,(40-21)$$

$$= 3.395321052$$

and, $\alpha = \frac{M}{2} = \frac{223}{2} = 111.5$

Also, $I_0(\beta) = I_0(3.395321052)$

$$= 1 + \frac{0.25\beta^2}{(1!)^2} + \frac{(0.25\beta^2)^2}{(2!)^2} + \frac{(0.25\beta^2)^3}{(3!)^2} + \ldots\ldots$$

$$= 6.621869$$

For a LPF, the ideal desired frequency response is given by

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & \text{for } -\omega_c \le \omega \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

and, $h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}\,d\omega$

or, $h_d[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\tau}\,e^{j\omega n}\,d\omega$

$-\omega_c$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega_c(n-\tau)}}{j(n-\tau)} - \frac{e^{-j\omega_c(n-\tau)}}{j(n-\tau)}\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega_c(n-\tau)} - e^{-j\omega_c(n-\tau)}}{j(n-\tau)}\right]$$

$$= \frac{1}{\pi(n-\tau)}\left[\frac{e^{j\omega_c(n-\tau)} - e^{-j\omega_c(n-\tau)}}{2j}\right]$$

$$\therefore\ h_d[n] = \frac{\sin[\omega_c(n-\tau)]}{\pi(n-\tau)} \text{ for } n \neq \tau$$

When $n = \tau$, $h_d[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega$

$$= \frac{1}{2\pi}[\omega]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi}[\omega_c - (-\omega_c)]$$

$$= \frac{1}{2\pi} \times 2\omega_c$$

$$= \frac{\omega_c}{\pi}$$

Therefore,

$$h_d[n] = \begin{cases} \frac{\sin[\omega_c(n-\tau)]}{\pi(n-\tau)}; & \text{for } n \neq \tau \\ \frac{\omega_c}{\pi}; & \text{for } n = \tau \end{cases}$$

We know,

$$\tau = \frac{\text{order} - 1}{2} = \frac{224-1}{2} = \frac{223}{2} = 111.5$$

$$h_d[n] = \begin{cases} \pi(n-111.5) & ; \text{ for } n \neq 111.5 \\ \frac{\omega_c}{\pi}; & \text{for } n = 111.5 \end{cases}$$

$$\Rightarrow h_d[n] = \begin{cases} \frac{\sin[0.2\pi(n-111.5)]}{\pi(n-111.5)}; & \text{for } n \neq 111.5 \\ \frac{0.2\pi}{\pi} = 0.2; & \text{for } n = 111.5 \end{cases}$$

$h[n] = h_d[n]\, w[n]$

For Kaiser window,

n	$h_d[n] = \frac{\sin[0.2\pi(n-111.5)]}{\pi(n-111.5)}$	$w[n] = \frac{I_0\left\{\beta\left[1-\left(\frac{n-\alpha}{\alpha}\right)^2\right]^{1/2}\right\}}{I_0(\beta)}$	$h[n] = h_d[n]w[n]$
0	2.3095×10^{-3}	0.15105	3.4885×10^{-4}
1	8.9016×10^{-4}	0.15889	1.41437×10^{-4}
2	-8.9829×10^{-4}	0.16689	-1.49915×10^{-4}
3	-2.3734×10^{-3}	0.17502	-4.154×10^{-4}
⋮	⋮	⋮	⋮
222	8.9016×10^{-3}	0.15889	1.41437×10^{-4}
223	2.30957×10^{-3}	0.15105	3.4885×10^{-4}

To calculate $\beta\left[1-\left(\frac{n-\alpha}{\alpha}\right)^2\right]^{1/2}$,

For n = 0, 0

n = 1, 0.4537143

n = 2, 0.6402021

n = 3, 0.7823083

⋮

$n = 222, 0.4537143$

$n = 223, 0$

Hence,

$\therefore$ h[n]

$= \{3.4885 \times 10^{-4}, 1.41437 \times 10^{-4}, -1.49915 \times 10^{-4}, -4.154 \times 10^{-4}, \ldots\ldots,$

$\uparrow$

$1.41437 \times 10^{-4}, 3.4885 \times 10^{-4}\}$

Example 6.3:

Kaiser window is to be used to design a linear phase FIR filter that meets following specifications

$$|H(e^{j\omega})| \leq 0.01, \quad 0.21\pi \leq |\omega| \leq \pi$$

$$0.95 \leq |H(e^{j\omega})| \leq 1.05, \quad 0 \leq |\omega| \leq 0.19\pi$$

Calculate the optimum value of ripple, attenuation and window length. [2080 Bhadra]

Solution:

Given specification is

$$|H(e^{j\omega})| \leq 0.01, \quad 0.21\pi \leq |\omega| \leq \pi$$

$$0.95 \leq |H(e^{j\omega})| \leq 1.05, \quad 0 \leq |\omega| \leq 0.19\pi$$

Comparing given specification with

$$1 - \delta_1 \leq |H(e^{j\omega})| \leq 1 + \delta_1, \quad \text{for } 0 \leq \omega \leq \omega_p$$

$$0 \leq |H(e^{j\omega})| \leq \delta_2, \quad \text{for } \omega_s \leq \omega \leq \pi$$

We have,

$\delta_1 = 0.05, \omega_p = 0.19\pi, \delta_2 = 0.01, \omega_s = 0.21\pi$

Now,

Optimum value of ripple $= \delta = \text{minimum }(\delta_1, \delta_2)$

$= \text{minimum }(0.05, 0.01)$

$= 0.01$

Attenuation, $A = -20\log_{10}\delta$

$= -20\log_{10}(0.01)$

$= 40 \text{ dB}$

Window length, $M = \dfrac{A - 8}{2.285\Delta\omega}$

$\Delta\omega = \omega_s - \omega_p$

$= 0.21\pi - 0.19\pi$

$= 0.02\pi$

So, $M = \dfrac{40 - 8}{2.285 \times 0.02\pi}$

$= 222.88657$

$\therefore$ $m \simeq 223$

Example 6.4:

Design a low pass FIR filter using suitable window to meet following specifications:

$$0.99 \leq |H(e^{j\omega})| \leq 1.01 \text{ for } 0 \leq |\omega| \leq 0.3\pi$$

$$|H(e^{j\omega})| \leq 0.01 \text{ for } 0.35\pi \leq |\omega| \leq \pi$$

[2081 Bhadra]

Solution:

$$0.99 \leq |H(e^{j\omega})| \leq 1.01 \text{ for } 0 \leq |\omega| \leq 0.3\pi$$

$$|H(e^{j\omega})| \leq 0.01 \text{ for } 0.35\pi \leq |\omega| \leq \pi$$

Comparing with,

$$1 - \delta_1 \leq |H(\omega)| \leq 1 + \delta_1 \text{ for } 0 \leq \omega \leq \omega_p$$

$$|H(\omega)| \leq \delta_2 \text{ for } \omega_s \leq \omega \leq \pi$$

We have,

$\delta_1 = 0.01, \omega_p = 0.3\pi,$

$\delta_2 = 0.01, \omega_s = 0.35\pi$

We know,

$\delta = \text{Minimum }(\delta_1, \delta_2) = 0.01$

$A = -20\log_{10}(\delta) = 40 \text{ dB}$

$M = \dfrac{A - 8}{2.285\Delta\omega}$; where, $\Delta\omega = \omega_s - \omega_p$

$= 0.35\pi - 0.3\pi$

$= 0.05\pi$

$$= \frac{40-8}{2.285 \times 0.05\pi}$$

$$= 89.1546 \simeq 90$$

Order of filter = M + 1 = 91

$$\omega_c = \frac{\omega_s + \omega_p}{2} = 0.325\pi$$

Here, for $21 \le A \le 50$, we have,

$$\beta = 0.5842\,(A-21)^{0.4} + 0.07886\,(A-21)$$

$$= 0.5842\,(40-21)^{0.4} + 0.07886\,(40-21)$$

$$= 3.395321052$$

$$\text{and, } \alpha = \frac{M}{2} = \frac{90}{2} = 45$$

Also,

$$I_o(\beta) = I_o(3.395321052)$$

$$= 1 + \frac{0.25\beta^2}{(1!)^2} + \frac{(0.25\beta)^2}{(2!)^2} + \frac{(0.25\beta^2)^3}{(3!)^2} + \ldots\ldots.$$

$$= 6.621869$$

For a LPF, the ideal desired frequency response is given by,

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & \text{for } -\omega_c \le \omega \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and, } h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega$$

$$\text{or, } h_d[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\tau} e^{j\omega n}\, d\omega$$

$$\text{or, } h_d[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\tau)}\, d\omega$$

$$\text{or, } h_d[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\tau)}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega_c(n-\tau)}}{j(n-\tau)} - \frac{e^{-j\omega_c(n-\tau)}}{j(n-\tau)}\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega_c(n-\tau)} - e^{-j\omega_c(n-\tau)}}{j(n-\tau)}\right]$$

$$= \frac{1}{\pi(n-\tau)}\left[\frac{e^{j\omega_c(n-\tau)} - e^{-j\omega_c(n-\tau)}}{2j}\right]$$

$$\therefore\ h_d[n] = \frac{\sin[\omega_c(n-\tau)]}{\pi(n-\tau)} \quad \text{for } n \ne \tau$$

$$\text{When } n = \tau,\ h_d[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega$$

$$= \frac{1}{2\pi}[\omega]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi}[\omega_c - (-\omega_c)]$$

$$= \frac{1}{2\pi} \times 2\omega_c = \frac{\omega_c}{\pi}$$

Therefore,

$$h_d[n] = \begin{cases} \dfrac{\sin[\omega_c(n-\tau)]}{\pi(n-\tau)} & ;\ \text{for } n \ne \tau \\ \dfrac{\omega_c}{\pi} & ;\ \text{for } n = \tau \end{cases}$$

We know,

$$\tau = \frac{\text{Order} - 1}{2} = \frac{91-1}{2} = 45$$

Hence,

$$h_d[n] = \begin{cases} \dfrac{\sin[0.325\pi(n-45)]}{\pi(n-45)} & ;\ \text{for } n \ne 45 \\ \dfrac{0.325\pi}{\pi} = 0.325 & ;\ \text{for } n = 45 \end{cases}$$

and, $h[n] = h_d[n]\ w[n]$

and, $w[n] = \dfrac{I_o\left\{\beta\left[1-\left(\frac{n-\alpha}{\alpha}\right)^2\right]^{\frac{1}{2}}\right\}}{I_o(\beta)}$ for $0 \le n \le M$

where, $I_o(\beta) = 6.621869$

and,

To calculate $\beta\left[1-.\left(\frac{n-\alpha}{\alpha}\right)^2\right]^{\frac{1}{2}}$,

For n = 0, 0

n = 1, 0.71181

n = 2, 1.00097

n = 3, 1.21896

n = 4, 1. 39942

.

.

.

n = 45, 3.395321

.

.

.

n = 89, 0.71181

n = 90, 0

Similarly,

n	w[n]
0	0.15101
1	0.17076
2	0.19127
3	0.21254
4	0.23449
. .	. .
.	.
45	1
. . .	. . .
89	0.17076
90	0.15101

For Kaiser window,

n	$h_d[n] = \dfrac{\sin[0.325\pi(n-45)]}{\pi(n-45)}$	w[n]	$h_d[n] = h_d[n]\ w[n]$
0	6.535×10^{-3}	0.15101	9.8685×10^{-4}
1	5.853×10^{-3}	0.17076	9.9945×10^{-4}
2	-5.808×10^{-4}	0.19127	-1.111×10^{-4}
3	-6.753×10^{-3}	0.21254	-1.435×10^{-3}
4	-6.620×10^{-3}	0.23449	-1.552×10^{-3}
. . .	. . .	. . .	. . .
45	0.325	1	0.325
. . .	. . .	. . .	. . .
89	5.853×10^{-3}	0.17076	9.9945×10^{-4}
90	6.535×10^{-3}	0.15101	9.8685×10^{-4}

Hence,

$h[n] =$

$\{9.8685\times10^{-4}, 9.9945\times10^{-4}, -1.111\times10^{-4}, -1.435\times10^{-3}, -1.552\times 10^{-3}, \ldots,$

$\uparrow$ $\qquad 0.325; \ldots, 9.9945\times10^{-4}, 9.8685\times10^{-4}\}$

Example 6.5:

Design the FIR filter using Kaiser window technique for the specifications:

$$0.899 \le |H(e^{j\omega})| \le 1 \qquad \text{for } |\omega| \le 0.2\pi$$

$$|H(e^{j\omega})| \le 0.01 \qquad \text{for } 0.4\pi \le \omega \le \pi$$

[2079 Bhadra, 2076 Chaitra]

Solution:

Given specification is

$$0.899 \le |H(e^{j\omega})| \le 1 \qquad \text{for } |\omega| \le 0.2\pi$$

$$|H(e^{j\omega})| \le 0.01 \qquad \text{for } 0.4\pi \le \omega \le \pi$$

Comparing given specification with

$$1 - \delta_1 \le |H(e^{j\omega})| \le 1 \qquad \text{for } |\omega| \le \omega_p$$

$$0 \le |H(e^{j\omega})| \le \delta_2 \qquad \text{for } \omega_s \le \omega \le \pi$$

We have,

$\delta_1 = 0.101,\ \omega_p = 0.2\pi,\ \delta_2 = 0.01,\ \omega_s = 0.4\pi$

We know,

$\delta = \text{minimum } (\delta_1, \delta_2) = 0.01$

$A = -20\log_{10}(\delta) = 40 \text{ dB}$

$$M = \frac{A-8}{2.285\Delta\omega}; \qquad \text{where } \Delta\omega = \omega_s - \omega_p$$

$$= 0.4\pi - 0.2\pi$$

$$= 0.2\pi$$

$$= \frac{40-8}{2.285 \times 0.2\pi}$$

$$= 22.28865 \simeq 23$$

Order of filter = M + 1 = 23 + 1 = 24

$$\omega_c = \frac{(\omega_s + \omega_p)}{2} = 0.3\pi$$

For $21 \le A \le 50$, we have

$$\beta = 0.5842(A-21)^{0.4} + 0.07886(A-21)$$

$$= 0.5842(40-21)^{0.4} + 0.07886(40-21)$$

$$= 3.395321052$$

and, $\alpha = \frac{M}{2} = \frac{23}{2} = 11.5$

Also,

$I_0(\beta) = I_0(3.395321052)$

$$= 1 + \frac{0.25\beta^2}{(1!)^2} + \frac{(0.25\beta^2)^2}{(2!)^2} + \frac{(0.25\beta^2)^3}{(3!)^2} + \ldots\ldots$$

$$= 6.621869$$

For a low pass FIR filter, the ideal desired frequency response is given by

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & \text{for } -\omega_c \le \omega \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

and, $h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}\, d\omega$

or, $h_d[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\tau}\, e^{j\omega n}\, d\omega$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\tau)}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega_c(n-\tau)} - e^{-j\omega_c(n-\tau)}}{j(n-\tau)}\right]$$

$$= \frac{1}{\pi(n-\tau)}\left[\frac{e^{j\omega_c(n-\tau)} - e^{-j\omega_c(n-\tau)}}{2j}\right]$$

$$\therefore\ h_d[n] = \frac{\sin[\omega_c(n-\tau)]}{\pi(n-\tau)} \quad \text{for } n \ne \tau$$

When $n = \tau$, $h_d[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega$

$$= \frac{1}{2\pi}[\omega]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi}[\omega_c - (-\omega_c)]$$

$$= \frac{2\omega_c}{2\pi}$$

$$= \frac{\omega_c}{\pi}$$

Therefore,

$$h_d[n] = \begin{cases} \frac{\sin[\omega_c(n-\tau)]}{\pi(n-\tau)}; & \text{for } n \neq \tau \\ \frac{\omega_c}{\pi}; & \text{for } n = \tau \end{cases}$$

We know,

$$\tau = \frac{\text{order} - 1}{2} = \frac{24-1}{2} = \frac{23}{2} = 11.5$$

$$\text{Hence, } h_d[n] = \begin{cases} \frac{\sin[0.3\pi(n-11.5)]}{\pi(n-11.5)}; & \text{for } n \neq 11.5 \\ \frac{0.3\pi}{\pi} = 0.3; & \text{for } n = 11.5 \end{cases}$$

$h[n] = h_d[n]\, w[n]$

For Kaiser window:

n	**$h_d[n] = \frac{\sin[0.3\pi(n-11.5)]}{\pi(n-11.5)}$**	**$w[n] = \frac{I_0\left\{\beta\left[1-\left(\frac{n-\alpha}{\alpha}\right)^2\right]^{1/2}\right\}}{I_0(\beta)}$**	**$h[n] = h_d[n]w[n]$**
0	–0.027338	0.15105	-4.1294×10^{-3}
1	–0.013762	0.232556	-3.2×10^{-3}
2	0.0152115	0.324080	4.9297×10^{-3}
3	0.036987	0.422398	0.01562323
.	.	.	.
.	.	.	.
.	.	.	.
22	–0.013762	0.232556	-3.2×10^{-3}
23	–0.027338	0.15105	-4.129×10^{-3}

To calculate $\beta\left[1-\left(\frac{n-\alpha}{\alpha}\right)^2\right]^{1/2}$,

For n = 0, 0

n = 1, 1.384823

n = 2, 1.913408

n = 3, 2.286960

.

.

.

n = 22, 1.384823

n = 23, 0

Hence,

$$h[n] = \{\underset{\uparrow}{-4.1294\times10^{-3}}, -3.2\times10^{-3}, 4.9297\times10^{-3}, 0.01562323, \ldots\ldots\ldots\ldots, -3.2\times10^{-3}; -4.1294\times10^{-3}\}$$

Example 6.6:

Design a low pass digital FIR filter having passband edge frequency $\omega_p = 0.2\pi$, stopband edge frequency $\omega_s = 0.45\pi$ and stopband attenuation α_s = 51 dB using any appropriate window function. [2079 Baishakh]

Solution:

Given specifications are:

Passband edge frequency, $\omega_p = 0.2\pi$

Stopband edge frequency, $\omega_s = 0.45\pi$

Stopband attenuation, α_s = 51 dB

The given stopband attenuation is closer to 53 dB.

So, we use Hamming window function.

For Hamming window,

$$w[n] = 0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right)$$

$$m = 3.3\frac{2\pi}{\Delta\omega}$$

Here,

$$\Delta\omega = \omega_s - \omega_p = 0.45\pi - 0.2\pi = 0.25\pi$$

Cut-off frequency, $\omega_c = \dfrac{\omega_s + \omega_p}{2}$

$$= \frac{0.45\pi + 0.2\pi}{2}$$

$$= 0.325\pi \text{ radian/sample}$$

So, $M = 3.3 \times \dfrac{2\pi}{0.25\pi} = 26.4 \simeq 27$

Length of filter (M) = 27

The desired frequency response for a low pass digital FIR filter is

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & ; |\omega| \le \omega_c \\ 0 & ; \text{elsewhere} \end{cases}$$

and, $h_d[n] = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}\, d\omega$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\tau} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\tau)}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega_c(n-\tau)} - e^{-j\omega_c(n-\tau)}}{j(n-\tau)}\right]$$

$$= \frac{1}{\pi(n-\tau)}\left[\frac{e^{j\omega_c(n-\tau)} - e^{-j\omega_c(n-\tau)}}{2j}\right]$$

$$= \frac{1}{\pi(n-\tau)}\sin[\omega_c(n-\tau)]$$

$$\therefore\ h_d[n] = \frac{\sin[\omega_c(n-\tau)]}{\pi(n-\tau)} \quad \text{for } n \ne \tau$$

When $n = \tau$, $h_d[n] = \dfrac{1}{2\pi}\displaystyle\int_{-\omega_c}^{\omega_c} d\omega$

$$= \frac{1}{2\pi}[\omega]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi}[\omega_c - (-\omega_c)]$$

$$= \frac{2\omega_c}{2\pi}$$

$$= \frac{\omega_c}{\pi}$$

$$= \frac{0.325\pi}{\pi}$$

$$= 0.325$$

Therefore,

$$h_d[n] = \begin{cases} \dfrac{\sin[\omega_c(n-\tau)]}{\pi(n-\tau)} & ; \text{for } n \ne \tau \\ 0.325 & ; \text{for } n = \tau \end{cases}$$

To find the value of τ:

For symmetric filter, we know

$h[n] = h[M-1-n]$

Also, $h[n] = h_d[n]\, w[n]$

So, $h_d[n]w[n] = h_d[M-1-n]w[n]$

$\Rightarrow h_d[n] = h_d[M-1-n]$

or, $\frac{\sin(n-\tau)}{\pi(n-\tau)} = \frac{\sin(M-1-n-\tau)}{\pi(M-1-n-\tau)}$

This above condition satisfies only if

$-(n-\tau) = M-1-n-\tau$

or, $-n+\tau = M-1-n-\tau$

or, $2\tau = M-1$

$\therefore\ \tau = \frac{M-1}{2}$

Since $m = 27$, $\tau = \frac{27-1}{2} = \frac{26}{2} = 13$

Hence,

$$h_d[n] = \begin{cases} \frac{\sin[0.325\pi(n-13)]}{\pi(n-13)} & ;\ \text{for } n \neq 13 \\ 0.325 & ;\ \text{for } n = 13 \end{cases}$$

n	$h_d[n] = \frac{\sin[0.325\pi(n-13)]}{\pi(n-13)}$	$w[n] = 0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right)$	$h[n] = h_d[n]w[n]$
0	0.015902	0.08	1.27216×10^{-3}
1	-8.196930×10^{-3}	0.093367	-7.6532×10^{-4}
2	−0.028138	0.132690	-3.7336×10^{-3}
3	−0.022508	0.195685	-4.4044×10^{-3}
. .	. .	. .	. .
13	0.325	1	0.325
. . .	. . .	. . .	. . .
25	-8.196930×10^{-3}	0.093367	-7.6532×10^{-4}
26	0.015902	0.08	1.27216×10^{-3}

Hence,

$\therefore$ h[n]

$$= \{\underset{\uparrow}{1.27216 \times 10^{-3}}, -7.6532 \times 10^{-4}, -3.7336 \times 10^{-3}, -4.4044 \times 10^{-3}, \ldots, 0.325, \ldots, -7.6532 \times 10^{-4}, 1.27216 \times 10^{-3}\}$$

Note: *If in the question, α_s = 41 dB (which is closer to 44 dB), we have to use Hanning window method. If in the question, α_s = 40 dB (which is closer to 44 dB), we have to use Hanning window method.*

Example 6.7:

Design a linear phase FIR filter using KAISER window to meet the following specifications.

$|H(e^{j\omega})| \leq 0.01;\ 0 \leq |\omega| \leq 0.25\pi$

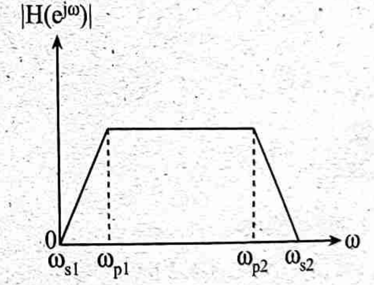
$0.95 \leq |H(e^{j\omega})| \leq 1.05;\ 0.35\pi \leq |\omega| \leq 0.6\pi$

$|H(e^{j\omega})| \leq 0.01;\ 0.65\pi \leq |\omega| \leq \pi$

[2078 Bhadra, 2074 Chaitra].

Solution:

The given specifications are of a band pass filter:



From the given specification, we have

$\omega_{s1} = 0.25\pi$, $\omega_{p1} = 0.35\pi$, $\omega_{s2} = 0.65\pi$, $\omega_{p2} = 0.6\pi$, $\delta_1 = 0.05$, $\delta_2 = 0.01$

We know,

$\delta = \text{minimum}\ (\delta_1, \delta_2)$

$= \text{minimum}\ (0.05, 0.01)$

$= 0.01$

$$A = -20\log_{10}(\delta) = 40 \text{ dB}$$

$$M = \frac{A-8}{2.285\Delta\omega}$$

Here,

$$\Delta\omega = \text{minimum }[(\omega_{p1} - \omega_{s1}), (\omega_{s2} - \omega_{p2})]$$

$$= \text{minimum }[(0.35\pi - 0.25\pi), (0.65\pi - 0.6\pi)]$$

$$= \text{minimum }[0.1\pi, 0.05\pi]$$

$$= 0.05\pi$$

$$\Rightarrow M = \frac{40-8}{2.285 \times 0.05\pi} = 89.15462895 \simeq 90$$

Order of the filter = M + 1 = 90 + 1 = 91

$$\alpha = \frac{M}{2} = \frac{90}{2} = 45$$

Also,

$$\omega_{c1} = \omega_{p1} - \frac{\Delta\omega}{2}$$

$$= 0.35\pi - \frac{0.05\pi}{2}$$

$$= 1.021017612$$

and,

$$\omega_{c2} = \omega_{p2} + \frac{\Delta\omega}{2}$$

$$= 0.6\pi + \frac{0.05\pi}{2}$$

$$= 1.963495408$$

Here, for $21 \le A \le 50$, we have

$$\beta = 0.5842(A-21)^{0.4} + 0.07886(A-21)$$

$$= 3.395321052$$

Using modified Bessel function, we have

$$I_0(\beta) = I_0(3.395321052)$$

$$= 1 + \frac{0.25\beta^2}{(1!)^2} + \frac{(0.25\beta^2)^2}{(2!)^2} + \frac{(0.25\beta^2)^3}{(3!)^2} + \text{........}$$

$$= 6.621869$$

For a band pass filter, the ideal desired frequency response is given by

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & \text{for } \omega_{c1} \le |\omega| \le \omega_{c2} \\ 0 & \text{otherwise} \end{cases}$$

So,

$$h_d[n] = \frac{1}{2\pi}\left[\int_{\omega_{c1}}^{\omega_{c2}} e^{-j\omega\tau} e^{j\omega n}\, d\omega + \int_{-\omega_{c2}}^{-\omega_{c1}} e^{-j\omega\tau} e^{j\omega n}\, d\omega\right]$$

$$= \frac{1}{2\pi}\left[\int_{\omega_{c1}}^{\omega_{c2}} e^{-j\omega(n-\tau)}\, d\omega + \int_{-\omega_{c2}}^{-\omega_{c1}} e^{-j\omega(n-\tau)}\, d\omega\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{\omega_{c1}}^{\omega_{c2}} + \left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{-\omega_{c2}}^{-\omega_{c1}}$$

$$= \frac{1}{2\pi}\left[\left[\frac{e^{j\omega_{c2}(n-\tau)} - e^{j\omega_{c1}(n-\tau)}}{j(n-\tau)}\right] + \left[\frac{e^{-j\omega_{c1}(n-\tau)} - e^{-j\omega_{c2}(n-\tau)}}{j(n-\tau)}\right]\right]$$

$$= \frac{1}{2\pi j(n-\tau)}\left[e^{j\omega_{c2}(n-\tau)} - e^{j\omega_{c1}(n-\tau)} + e^{-j\omega_{c1}(n-\tau)} - e^{-j\omega_{c2}(n-\tau)}\right]$$

$$= \frac{1}{\pi(n-\tau)}\left[\frac{e^{j\omega_{c2}(n-\tau)} - e^{-j\omega_{c2}(n-\tau)}}{2j} - \left\{\frac{e^{j\omega_{c1}(n-\tau)} - e^{-j\omega_{c1}(n-\tau)}}{2j}\right\}\right]$$

$$= \frac{1}{\pi(n-\tau)}[\sin(\omega_{c2}(n-\tau)) - \sin(\omega_{c1}(n-\tau))]$$

$$\therefore\ h_d[n] = \frac{\sin(\omega_{c2}(n-\tau)) - \sin(\omega_{c1}(n-\tau))}{\pi(n-\tau)} \text{ for } n \ne \tau$$

$$\text{When } n = \tau,\ h_d[n] = \frac{1}{2\pi}\left[\int_{\omega_{c1}}^{\omega_{c2}} d\omega + \int_{-\omega_{c2}}^{-\omega_{c1}} d\omega\right]$$

$$= \frac{1}{2\pi}\left[[\omega]_{\omega_{c1}}^{\omega_{c2}} + [\omega]_{-\omega_{c2}}^{-\omega_{c1}}\right]$$

$$= \frac{1}{2\pi}[\omega_{c2} - \omega_{c1} + (-\omega_{c1}) - (-\omega_{c23})]$$

$$= \frac{2\omega_{c2} - 2\omega_{c1}}{2\pi} = \frac{\omega_{c2} - \omega_{c1}}{\pi}$$

Therefore,

$$h_d[n] = \begin{cases} \dfrac{\sin[\omega_{c2}(n-\tau)] - \sin[\omega_{c1}(n-\tau)]}{\pi(n-\tau)} & ; \text{for } n \neq \tau \\ \dfrac{\omega_{c2}-\omega_{c1}}{\pi} & ; \text{for } n = \tau \end{cases}$$

We know,

$$\tau = \frac{\text{Order} - 1}{2} = \frac{91-1}{2} = \frac{90}{2} = 45$$

Hence,

$$h_d[n] = \begin{cases} \dfrac{\sin[1.9635(n-45)] - \sin[1.0210(n-45)]}{\pi(n-45)} & ; \text{for } n \neq 45 \\ 0.3 & ; \text{for } n = 45 \end{cases}$$

$h[n] = h_d[n]w[n]$

For Kaiser window, the window function w[n] is

$$w[n] = \frac{I_0\left\{\beta\left[1-\left(\dfrac{n-\alpha}{\alpha}\right)^2\right]^{1/2}\right\}}{I_0(\beta)}$$

So,

n	$h_d[n]$	w[n]	h[n]
0	-3.82897×10^{-3}	0.15101	-5.7821×10^{-4}
1	–0.01308	0.17076	-2.2335×10^{-3}
2	3.41787×10^{-3}	0.191276	6.5376×10^{-4}
3	0.012115	0.21254	2.5749×10^{-3}
. . .	. . .	. . .	. . .
45	0.3	1	0.3
. .	. .	. .	. .
89	–0.01308	0.17076	-2.2335×10^{-3}
90	-3.82897×10^{-3}	0.15101	-5.7821×10^{-4}

Hence,

$\therefore$ h[n]

$$= \{\underset{\uparrow}{-5.7821} \times 10^{-4}, -2.2335 \times 10^{-3}, 6.5376 \times 10^{-4}, 2.5749 \times 10^{-3}, \ldots\ldots,$$
$$0.3, \ldots\ldots, -2.2335\times10^{-3}, -5.7821 \times 10^{-4}\}$$

Q. Why is Kaiser window better than other fixed windows in FIR filter design? [2074 Ashwin, 2072 Chaitra]

Solution:

Kaiser window offers several advantages over other windows in digital filter design.

i. Adjustable parameter (β): The Kaiser window has an adjustable parameter (β) that allows controlling the trade-off between the main lobe width and stopband attenuation. This flexibility makes it adaptable to different design requirements.

ii. By adjusting β, the side lobe levels can be minimized, providing better stopband attenuation compared to fixed windows like the Hamming or Hanning windows.

iii. The Kaiser window can achieve high stopband attenuation, making it suitable for applications requiring suppression of unwanted frequencies.

5.3 Filter Design by Frequency Sampling Method

Suppose we want to design a FIR filter whose desired frequency response is $H_d(\omega)$. This frequency response is sampled uniformly at M points. Such frequency samples are given by,

$$\omega_k = \frac{2\pi}{M}k; \text{ where } k = 0, 1, 2, \ldots\ldots, M-1$$

Such sampled frequency response is called Discrete Fourier Transform and is denoted by H(k) i.e.

$$H(k) = H_d(\omega)|_{\omega=\omega k} \qquad ; k = 0, 1, 2, \ldots\ldots, M-1$$

$$\text{or, } H(k) = H_d\left(\frac{2\pi}{M}k\right) \qquad ; k = 0, 1, 2, \ldots\ldots, M-1$$

Here, H(k) is called M point DFT. We can obtain h[n] by taking inverse discrete Fourier transform of H(k). This h[n] is called unit sample response of FIR filter. i.e.,

$$h[n] = \frac{1}{M}\sum_{k=0}^{M-1} H(k)\, e^{j2\pi kn/M} \;;\; n = 0, 1, \ldots\ldots, M-1 \ldots\ldots (I)$$

Hence, the unit sample response h(n) of FIR filter of length 'M' is obtained using frequency sampling technique.

For FIR filter to be realizable, the coefficients of h[n] should be all real. This is possible if all complex terms appear in complex conjugate pairs.

Let us consider the term $H(M-k)e^{j2\pi n(M-k)/M}$. It can be written as,

$$H(M-k)e^{-j2\pi kn/M}.e^{j2\pi n}$$

We know,

$$e^{j2\pi n} = \cos 2\pi n + j\sin 2\pi n = 1$$

$$\therefore H(M-k)e^{-j2\pi(M-k)/M} = H(M-k)e^{-j2\pi kn/M}$$

From the periodicity of DFT,

$$|H(M-k)| = |H(k)|$$

$$\text{So, } H(M-k)e^{-j2\pi kn/M} = H(k)e^{-j2\pi kn/M}$$

The term $H(k)e^{-j2\pi kn/M}$ is a complex conjugate of $H(k)e^{j2\pi kn/M}$. Hence, $H(M-k)e^{-j2\pi kn/M}$ is also a complex conjugate of $H(k)e^{j2\pi kn/M}$.

$$\Rightarrow H(M-k) = H^*(k)$$

Using this relationship of complex conjugate, we can write the term of unit sample response h[n] as

$$h[n] = \left\{H(0) + 2\sum_{k=1}^{P} P_e[H(k)e^{j2\pi kn/M}]\right\}$$

where,

$$P = \begin{cases} \frac{M-1}{2} & \text{if M is odd} \\ \frac{M}{2} - 1 & \text{if M is even} \end{cases}$$

This equation is obtained by combining complex conjugate terms in equation (I) and the above expression may be used to compute coefficients of FIR filter.

Summarized:

$$H_d(e^{j\omega}) \text{ or } H_d(\omega) \xrightarrow{\text{M-points}} H(k) \xrightarrow{\text{IDFT}} h[n]$$

5.4 Filter Design using Optimum Approximation, Remez Exchange Algorithm

Window method and frequency sampling methods are relatively simple technique for designing linear phase FIR filters. However, they contain some minor disadvantages which may make it undesirable for certain applications. One of the major disadvantage is the inability to precisely control the critical frequencies such ω_p and ω_s and deviation between the required frequency response and actual frequency response.

Optimal Filter Method

An *optimal filter method* is a technique, used to design filters that best meet the specified criteria, in which weighted approximation error between the desired frequency response and actual frequency response is spread evenly across the passband and stopband of the filter, minimizing the maximum error.

Consider the design of a low pass filter with passband edge frequency ω_p and stopband edge frequency ω_s..

$$1 - \delta_1 \le H_r(\omega) \le 1 + \delta_1 \qquad ; |\omega| \le \omega_p$$

$$0 \le H_r(\omega) \le \delta_2 \qquad ; |\omega| > \omega_s$$

Here, δ_1 and δ_2 represents the ripple in passband and stopband.

Let $H_r(\omega)$ be the real valued frequency response of h[n]. We can represent $H_r(\omega)$ in the form:

$$H_r(\omega) = Q(\omega)P(\omega)$$

where,

$$Q(\omega)=\begin{cases}1 & ;\text{Case 1}\\ \cos\frac{\omega}{2} & ;\text{Case 2}\\ \sin\omega & ;\text{Case 3}\\ \sin\frac{\omega}{2} & ;\text{Case 4}\end{cases}$$

and,

$$P(\omega)=\sum_{k=0}^{L}\alpha(k)\cos(\omega k)\qquad\begin{array}{l};L=(M-1)/2\text{ for case 1}\\ ;L=(M-3)/1\text{ for case 3}\\ ;L=(M/2)-1\text{ for case 2 and 4}\end{array}$$

Here;

Case 1: Symmetric unit sample response $h[n]=h[M-1-n]$ and M odd.

Case 2: Symmetric unit sample response $h[n]=h[M-1-n]$ and M even.

Case 3: Antisymmetric unit sample response $h[n]=-h[M-1-n]$ and M odd.

Case 4: Antisymmetric unit sample response $h[n]=-h[M-1-n]$ and M even.

Summarized: Real Valued Frequency Response Functions for Linear Phase FIR Filter

Filter type	Q(ω)	P(ω)
$h[n]=h[M-1-n]$, M odd; Case 1	1	$\sum_{k=0}^{(M-1)/2} a(k)\cos\omega k$
$h[n]=h[M-1-n]$, M even; Case 2	$\cos\left(\frac{\omega}{2}\right)$	$\sum_{k=0}^{(M/2)-1} \tilde{b}(k)\cos\omega k$
$h[n]=-h[M-1-n]$, M odd; Case 3	$\sin(\omega)$	$\sum_{k=0}^{(M-3)/2} \tilde{c}(k)\cos\omega k$
$h[n]=-h[M-1-n]$, M even; Case 4	$\sin\left(\frac{\omega}{2}\right)$	$\sum_{k=0}^{(M/2)-1} \tilde{d}(k)\cos\omega k$

Let the real valued desired frequency response be $H_{dr}(\omega)$ and the weighting function on the approximation error be $W(\omega)$.

Now, the weighted approximation error is given as,

$$E(\omega)=W(\omega)[H_{dr}(\omega)-H_r(\omega)]$$

$$=W(\omega)[H_{dr}(\omega)-Q(\omega)P(\omega)]$$

$$=W(\omega)Q(\omega)\left[\frac{H_{dr}(\omega)}{Q(\omega)}-P(\omega)\right]$$

$$\therefore E(\omega)=\tilde{W}(\omega)\,[\tilde{H}_{dr}(\omega)-P(\omega)]$$

where,

$$\tilde{W}(\omega)=W(\omega)Q(\omega)$$

$$\tilde{H}_{dr}(\omega)=\frac{H_{dr}(\omega)}{Q(\omega)}$$

for all four different types of linear-phase FIR filters.

Given the error function $E(\omega)$, the Chebyshev approximation problem is basically to determine the filter parameter $\{\alpha(k)\}$ that minimize the maximum absolute value of $E(\omega)$ over the frequency bands in which the approximation is to be performed.

The solution to this problem is found using alternation theorem:

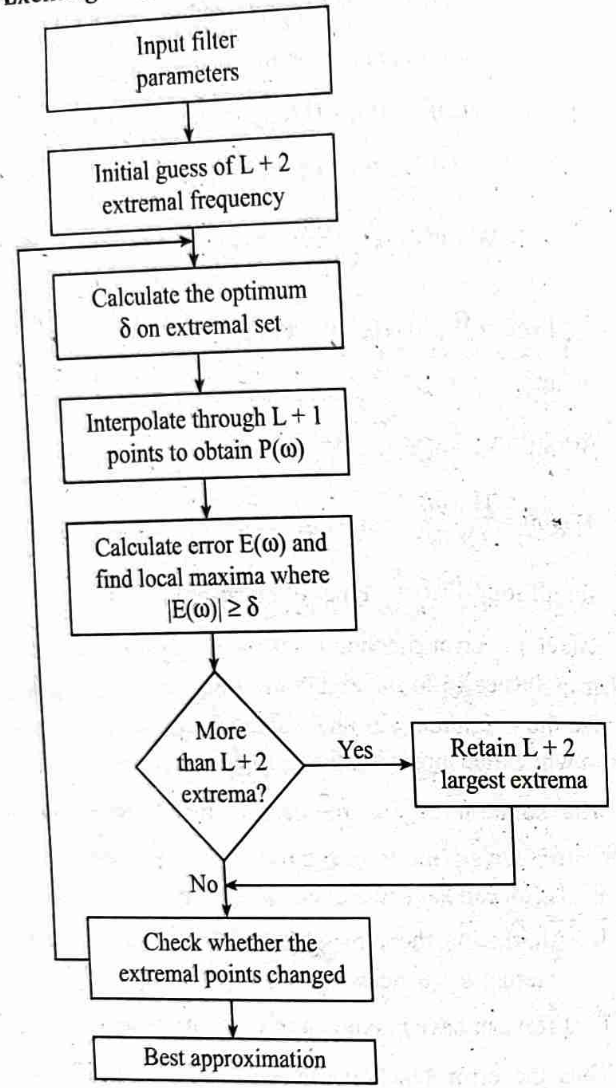
i. $H_r(\omega)$ can have maximum of L + 1 maxima or minima.

ii. Alternation theorem further states that there are L + 2 external frequencies in $E(\omega)$.

iii. $E(\omega)$ can have maximum of L + 3 maxima or minima.

Thus the error function can have either L + 2 or L + 3 extrema. If L + 3 alternations are present, it is called maximum ripple filter.

Remez Exchange Algorithm:

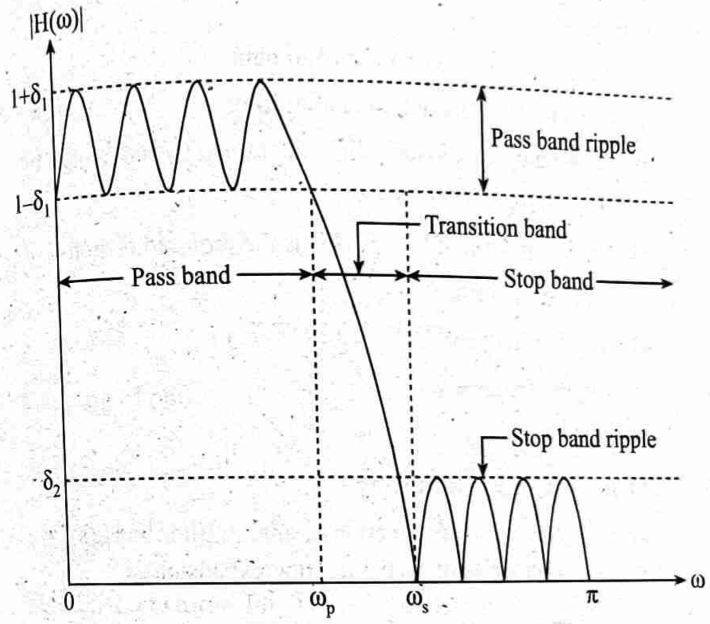


Chapter – 6

IIR FILTER DESIGN

Analog filter design is a matured and well developed field, so we begin the design of IIR digital filter in the analog domain first and then convert the design into digital domain.

Magnitude Characteristics



δ_1 =Passband ripple

δ_2 = Stopband ripple

ω_p = Passband edge frequency

ω_s = Stopband edge frequency

6.1 Filter Design by Impulse Invariance Method

Design steps:

I. The transfer function of analog filter $H_a(s)$ is usually given or obtained from the given specifications.

II. $H_a(s)$ is expanded using partial fraction expansion if required.

III. Obtain z-transform of each partial fraction expansion term using impulse invariance transformation equation.

IV. Combine all the z-transforms to obtain H(z), which is the required digital IIR filter.

In the impulse invariance method, we design an analog filter from the given specification and convert the analog filter to digital filter.

Notations used:

$h_a(t)$ = Impulse response in time domain

$H_a(s)$ = Transfer function of analog filter

$h_a(nT)$ = Sampled version of $h_a(t)$, obtained by replacing t by nT

$H(z)$ = z-transform of $h(nT)$. This is the required response of digital filter

Ω = Analog frequency

ω = Digital frequency

And, $\omega = \Omega t$

Transformation of $H_a(s)$ to $H(z)$

Let the system transfer function of analog filter be $H_a(s)$. We can express $H_a(s)$ in terms of partial fraction expansion as

$$H_a(s) = \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \frac{A_3}{s-p_3} + \ldots\ldots$$

$$= \sum_{i=1}^{N} \frac{A_i}{s-p_i} \quad \ldots\ldots \text{(i)}$$

Here, $A_i = A_1, A_2, A_3, \ldots\ldots, A_N$ are coefficients of partial fraction expansion.

$p_i = p_1, p_2, p_3, \ldots\ldots, p_N$ are the poles

Taking inverse Laplace transformation of $H_a(s)$ in equation (i),

$$h_a(t) = \sum_{i=1}^{N} A_i e^{p_i t}$$

Sampling $h_a(t)$ (i.e. replacing t by nT), we get

$$h[n] = \sum_{i=1}^{N} A_i e^{p_i nT}; \text{ T is sampling interval}$$

Taking z-transform of h[n], we get

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$$

Substituting the value of h[n], we get

$$H(z) = \sum_{n=0}^{\infty} \left[\sum_{i=1}^{N} A_i e^{p_i nT} \right] z^{-n}$$

$$\text{or, } H(z) = \sum_{i=1}^{N} A_i \sum_{n=0}^{\infty} (e^{p_i T} z^{-1})^n$$

Using, $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$, we get

$$H(z) = \sum_{i=1}^{N} A_i \left(\frac{1}{1 - e^{p_i T} z^{-1}} \right)$$

Mapping of poles:

$$H_a(s) = \sum_{i=1}^{N} \frac{A_i}{s-p_i}, \quad H(z) = \sum_{i=1}^{N} A_i \left(\frac{1}{1 - e^{p_i T} z^{-1}} \right)$$

$$\text{So, } \frac{1}{s-p_i} \rightarrow \frac{1}{1 - e^{p_i T} z^{-1}}$$

Standard Formula for Transformation in Impulse Invariance Method (IIM)

I. $\frac{1}{s-p_i} \rightarrow \frac{1}{1 - e^{p_i T} z^{-1}}$

II. $\frac{s+a}{(s+a)^2 + b^2} \rightarrow \frac{1 - e^{-aT}[\cos bT] z^{-1}}{1 - 2e^{-aT}[\cos bT] z^{-1} + e^{-2aT} z^{-2}}$

III. $\frac{b}{(s+a)^2 + b^2} \rightarrow \frac{e^{-aT} \sin(bT) z^{-1}}{1 - 2e^{-aT}(\cos bT) z^{-1} + e^{-2aT} z^{-2}}$

Example 6.1:

Determine H(z) using impulse invariance method at 5Hz sampling frequency from given $H_a(s) = \frac{2}{(s+1)(s+2)}$.

Solution:

Using partial fraction,

$$H_a(s) = \frac{2}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} = \frac{As + 2A + Bs + B}{(s+1)\,(s+2)}$$

$$\text{or, } H_a(s) = \frac{2}{(s+1)(s+2)} = \frac{(A+B)s + 2A + B}{(s+1)(s+2)}$$

Comparing coefficients, we get

$A + B = 0$

and, $2A + B = 2$

On solving,

$A = 2, B = -2$

$$\text{So, } H_a(s) = \frac{2}{(s+1)} - \frac{2}{(s+2)}$$

Comparing with $\frac{1}{s-p_i} \rightarrow \frac{1}{1-e^{p_iT}z^{-1}}$, we have

$$T = \frac{1}{F_s} = \frac{1}{5} = 0.2 \text{ sec}$$

Poles at $p_1 = -1$ and $p_2 = -2$

$$\text{Now, } \frac{1}{s+1} \rightarrow \frac{1}{1-e^{-1\times 0.2}z^{-1}} = \frac{1}{1-e^{-0.2}z^{-1}}$$

$$\frac{1}{s+2} \rightarrow \frac{1}{1-e^{-2\times 0.2}z^{-1}} = \frac{1}{1-e^{-0.4}z^{-1}}$$

Therefore,

$$H(z) = \frac{2}{1-e^{-0.2}z^{-1}} - \frac{2}{1-e^{-0.4}z^{-1}}$$

$$= \frac{2}{1-0.8187z^{-1}} - \frac{2}{1-0.67z^{-1}}$$

$$= \frac{2z}{z-0.8187} - \frac{2z}{z-0.67}$$

$$= \frac{2z^2 - 1.36z - 2z^2 + 1.6374z}{(z-0.8187)\,(z-0.67)}$$

$$\therefore\ H(z) = \frac{0.2774z}{z^2 - 1.4887z + 0.54853}$$

6.2 Filter Design Using Bilinear Transformation

The IIR filter design using Impulse Invariance Method (IIM) has a severe limitation in that they are only for lowpass filters and a limited class of bandpass filters.

The bilinear transformation describes a mapping from the s-plane to the z-plane. It overcomes the limitation of IIM. It is a conformal mapping that transforms the jΩ-axis into the unit circle in the z-plane only once, thus avoiding aliasing of frequency components.

Let us consider an analog filter with system function

$$H_a(s) = \frac{b}{s+a} \text{ .... (i)}$$

We can obtain differential equation which describes the analog filter as

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

or, $sY(s) + aY(s) = bX(s)$

Taking inverse Laplace transform on both sides,

$$\frac{dy(t)}{dt} + ay(t) = bx(t) \text{ .... (ii)}$$

Integrating equation (ii) between (nT – T) and nT i.e.

$$\int_{nT-T}^{nT} \frac{dy(t)}{dt}\,dt + a\int_{nT-T}^{nT} y(t)dt = b\int_{nT-T}^{nT} x(t)\,dt \text{ .... (iii)}$$

The trapezoidal rule for numeric integration is

$$\int_{nT-T}^{nT} a(t)dt = \frac{T}{2}\,[a(nT) + a(nT-T)]$$

Here, equation (iii) becomes

$$y(nT) - y(nT-T) + \frac{aT}{2}\,y(nT) + \frac{aT}{2}\,y(nT-T)$$

$$= \frac{bT}{2}\,x(nT) + \frac{bT}{2}\,x(nT-T)$$

Taking z-transform, the system function of the digital filter will be

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a} \;\; .... \; (iv)$$

Comparing equation (i) and (iv), we get

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$\therefore s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

Relation between analog and digital frequency:

$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$ is *analog frequency*

$\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$ is *digital frequency*

Advantages of bilinear transformation over impulse invariance method:

i. The frequency mapping is one-to-one that prevents aliasing.

ii. Stable analog filter is transformed into stable digital filter (preserves stability).

iii. Maintains a consistent frequency response.

iv. Easy and effective to implement.

Comparison between IIM and Bilinear Transformation

Impulse Invariance Method	Bilinear Transformation Method
i. Poles are transferred by using the expression $\frac{1}{s-p_i} \rightarrow \frac{1}{1-e^{p_i T}z^{-1}}$	i. Poles are transferred by using the expression $s = \frac{2}{T}\left[\frac{z-1}{z+1}\right]$
ii. It is prone to aliasing effect.	ii. It avoids aliasing effect by frequency warping.
iii. Mapping is many to one.	iii. Mapping is one to one.
iv. It is not suitable to design HPF and BPF.	iv. HPF and BPF can be designed.
v. Only poles of the system can be mapped.	v. Poles as well as zeros can be mapped.
vi. More complex due to aliasing issues.	vi. Simple and efficient to implement.

Example 6.2:

The transfer function of analog filter is $H_a(s) = \frac{3}{(s+2)(s+3)}$ with T = 0.1sec. Design digital IIR filter using BLT.

Solution:

Given, $H_a(s) = \frac{3}{(s+2)(s+3)}$ and T = 0.1 sec

Putting $s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$, we get

$$s = \frac{2}{0.1}\left(\frac{z-1}{z+1}\right) = 20\left(\frac{z-1}{z+1}\right)$$

$$H(z) = \frac{3}{\left[20\left(\frac{z-1}{z+1}\right)+2\right]\left[20\left(\frac{z-1}{z+1}\right)+3\right]}$$

$$= \frac{3}{\left[\frac{(20z-20+2z+2)}{(z+1)}\right]\left[\frac{(20z-20+3z+3)}{(z+1)}\right]}$$

$$= \frac{3(z+1)^2}{(22z-18)\,(23z-17)}$$

$$= \frac{3(z^2+2z+1)}{506z^2-374z-414z+306}$$

$$= \frac{3(z^2+2z+1)}{506z^2-788z+306}$$

$$= \frac{z^2+2z+1}{168.67z^2-262.67z+102}$$

Conversion of Low Pass filter to High Pass filter

Put $z^{-1} = -\left[\frac{z^{-1}+\alpha}{1+\alpha z^{-1}}\right]$

where, $\alpha = -\frac{\cos[(\omega_p' + \omega_p)/2]}{\cos[(\omega_p' - \omega_p)/2]}$

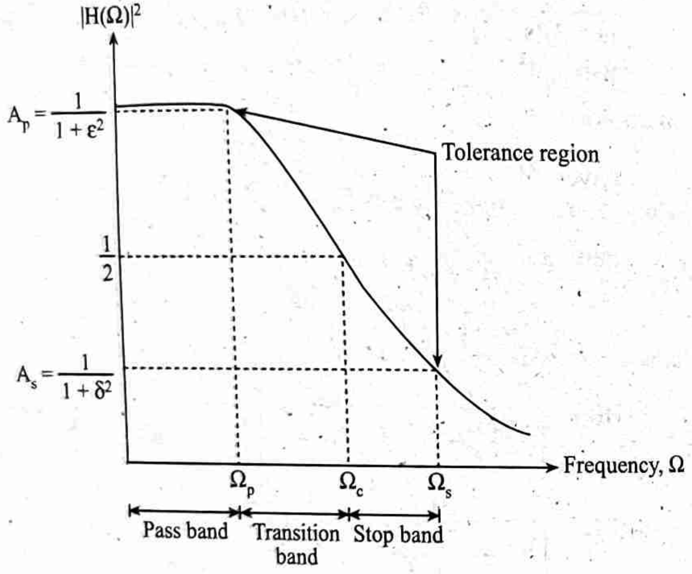
Here,

ω_p = Passband frequency of LPF

ω_p' = Passband frequency of HPF

6.3 Design of Digital Low Pass Butterworth Filter

Butterworth Approximation



The magnitude squared response of low pass Butterworth filter is

$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$$= \frac{1}{1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}}$$

where,

$|H(\Omega)|$ = Magnitude of analog low pass filter (LPF)

Ω_c = Cut-off frequency (–3dB frequency)

Ω_p = Passband edge frequency

$1+\varepsilon^2$ = Passband edge value

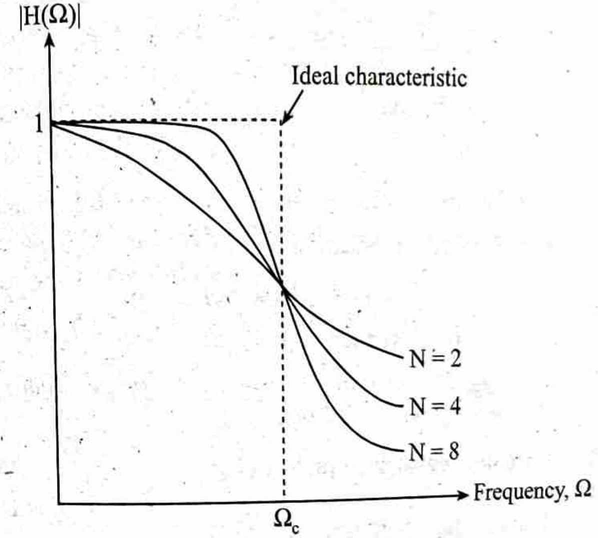
$1+\delta^2$ = Stopband edge value

ε = Parameters related to ripples in pass band

δ = Parameters related to ripples in stop band

N = Order of the filter

Higher value of N gives better filter approximations but is more complex and expensive.



Design steps:

I. Obtain the equivalent analog filter to be designed for the given specification of digital filter:

For impulse invariance: $\Omega = \frac{\omega}{T}$

For bilinear transformation: $\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$

where,

Ω = Frequency of analog filter (rad/sec)

ω = Frequency of digital filter (rad/sample)

T = Sampling time

II. **Evaluate the order of the filter:**

$$N = \frac{\log_{10}\left[\dfrac{\frac{1}{A_s^2} - 1}{\frac{1}{A_p^2} - 1}\right]}{2\log_{10}\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

In dB,

$$N = \frac{\log_{10}\left[\dfrac{10^{0.1A_s(dB)} - 1}{10^{0.1A_p(dB)} - 1}\right]}{2\log_{10}\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

where,

A_p = Passband attenuation

A_s = Stopband attenuation

Ω_s = Stopband frequency (rad/sec)

Ω_p = Passband frequency (rad/sec)

Note: *While calculating order, take (upper limit) ceiling value: E.g.: 1.07 = 2, 1.0001 = 2.*

III. **Calculate cut-off frequency (Ω_c):**

For impulse invariance: $\Omega_c = \dfrac{\omega_c}{T}$

For bilinear transformation: $\Omega_c = \dfrac{2}{T}\tan\left(\dfrac{\omega_c}{2}\right)$

If cut-off frequency (ω_c) is not given, then

$$\Omega_c = \frac{\Omega_p}{\left(\frac{1}{A_p^2} - 1\right)^{\frac{1}{2N}}}$$

In dB,

$$\Omega_c = \frac{\Omega_p}{(10^{0.1A_p(dB)} - 1)^{\frac{1}{2N}}}$$

Note: *$A_p(dB)$ and $A_s(dB)$ can be denoted by α_p and α_s respectively.*

IV. **Determine pole locations:**

$p_i = \pm\Omega_c e^{j(N+2i+1)\pi/2N}$; i = 0, 1, 2, ..., N – 1

V. **Calculate the transfer function:**

$$H_a(s) = \frac{\Omega_c^{\ N}}{(s - p_0)(s - p_1) \ldots\ldots\ldots (s - p_{N-1})}$$

Note: *If poles of a system are on left hand side of z-plane, then only the system in stable.*

VI. Finally we design the digital filter using Bilinear Transformation Method or Impulse Invariance Method. Consider only stable poles.

Example 6.3:

Using bilinear transformation, design a digital filter using Butterworth approximation which satisfies the following conditions.

$0.8 \le |H(e^{j\omega})| \le 1$ for $0 \le \omega \le 0.2\pi$

$|H(e^{j\omega})| \le 0.2$ for $0.6\pi \le \omega \le \pi$

[2075 Chaitra, 2070 Chaitra]

Solution:

Given,

$0.8 \le |H(e^{j\omega})| \le 1$ for $0 \le \omega \le 0.2\pi$

$|H(e^{j\omega})| \le 0.2$ for $0.6\pi \le \omega \le \pi$

From the question, the digital filter specifications are:

$A_p = 0.8$, $\omega_p = 0.2\pi$ rad/sample

$A_s = 0.2$, $\omega_s = 0.6\pi$ rad/sample

Assume T = 1 sec,

we have

I. Equivalent analog filter using bilinear transformation is:

$$\Omega_p = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right) = 2\tan\left(\frac{0.2\pi}{2}\right)$$

$$= 0.649839 \simeq 0.65 \text{ rad/sec}$$

and,

$$\Omega_s = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right) = 2\tan\left(\frac{0.6\pi}{2}\right) = 2.75276 \text{ rad/sec}$$

Required analog filter specifications are

$A_p = 0.8, \Omega_p = 0.65$ rad/sec,

$A_s = 0.2, \Omega_s = 2.75276$ rad/sec

II. Calculating order of the filter:

$$N = \frac{\log\left[\frac{\frac{1}{A_s^2}-1}{\frac{1}{A_p^2}-1}\right]}{2\log\left(\frac{\Omega_s}{\Omega_p}\right)} = \frac{\log\left(\frac{24}{0.5625}\right)}{2\log\left(\frac{2.7527}{0.65}\right)} = 1.3$$

$\therefore\ N \simeq 2$

III. Calculating cut-off frequency since ω_c is not given:

$$\Omega_c = \frac{\Omega_p}{\left(\frac{1}{A_p^2}-1\right)^{\frac{1}{2N}}} = \frac{0.65}{\left(\frac{1}{0.8^2}-1\right)^{\frac{1}{2\times2}}}$$

$\therefore\ \Omega_c = 0.75055$ rad/sec

IV. Determining poles of $H_a(s)$:

$p_i = \pm\Omega_c e^{j(N+2i+1)\pi/2N}$; $i = 0, 1, \ldots\ldots, N-1$

For $i = 0$,

$$p_0 = \pm 0.75e^{j(2+1)\pi/4}$$

$$= \pm 0.75e^{3\pi j/4}$$

$$= \pm 0.75\left[\cos\left(\frac{3\pi}{4}\right) + j\sin\left(\frac{3\pi}{4}\right)\right]$$

$$= \pm(-0.53 + j0.53)$$

$\therefore\ p_0 = -0.53 + j0.53$ and $0.53 - j0.53$

For $i = 1$,

$$p_1 = \pm 0.75e^{j(2+2+1)\pi/4}$$

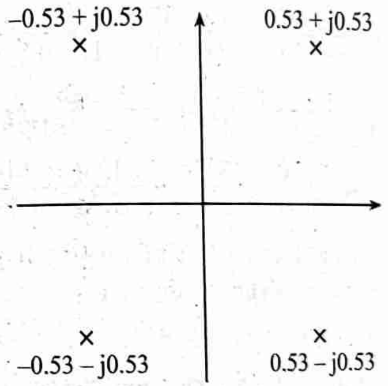
$$= \pm 0.75e^{j5\pi/4}$$

$$= \pm 0.75\left[\cos\left(\frac{5\pi}{4}\right) + j\sin\left(\frac{5\pi}{4}\right)\right]$$

$$= \pm[(-0.53) + j(-0.53)]$$

$\therefore\ p_1 = -0.53 - j0.53$ and $0.53 + j0.53$

Plotting the poles, we get



Taking stable poles only, we get

$p_0 = -0.53 + j0.53$

$p_1 = -0.53 - j0.53$

V. Calculating $H_a(s)$:

$$H_a(s) = \frac{\Omega_c^N}{(s-p_0)(s-p_1)}$$

$$= \frac{0.75055^2}{(s+0.53-j0.53)(s+0.53+j0.53)}$$

$$= \frac{0.563325}{s^2+0.53s+j0.53s+0.53s+0.2809+j0.2809-j0.53s-j0.2809-j^2 0.2809}$$

$$= \frac{0.563325}{s^2+1.06s+0.2809+0.2809}$$

$$= \frac{0.563325}{s^2+1.06s+0.5618}$$

VI. Calculating H(z) using bilinear transformation:

Put $s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$ in $H_a(s)$

or, $s = 2\left(\frac{z-1}{z+1}\right)$, then

$$H(z) = \frac{0.563325}{\left\{2\left(\frac{z-1}{z+1}\right)\right\}^2 + 1.06 \times 2\left(\frac{z-1}{z+1}\right) + 0.5618}$$

$$= \frac{0.563325}{4\frac{(z-1)^2}{(z+1)^2} + 2.12\left(\frac{z-1}{z+1}\right) + 0.5618}$$

$$= \frac{0.563325(z+1)^2}{4(z-1)^2 + 2.12(z-1)(z+1) + 0.5618(z+1)^2}$$

$$= \frac{0.563325z^2 + 1.1265z + 0.563325}{4z^2 - 8z + 4 + 2.12z^2 - 2.12 + 0.5618z^2 + 1.1236z + 0.5618}$$

$$\therefore \; H(z) = \frac{0.563325z^2 + 1.12665z + 0.563325}{6.6818z^2 - 6.8764z + 2.4418}$$

which is the transfer function of the required digital filter using Butterworth approximation.

Example 6.4:

Using Bilinear transformation, design a Butterworth low pass filter which satisfies the following conditions

$$0.9 \le |H(e^{j\omega})| \le 1, \qquad \text{for } 0 \le \omega \le \frac{\pi}{2}$$

$$|H(e^{j\omega})| \le 0.2, \qquad \text{for } \frac{3\pi}{4} \le \omega \le \pi$$

Consider sampling frequency of 1 Hz.

[2081 Bhadra, 2080 Bhadra]

Solution:

Given,

$$0.9 \le |H(e^{j\omega})| \le 1 \;; \qquad \text{for } 0 \le \omega \le \frac{\pi}{2}$$

$$|H(e^{j\omega})| \le 0.2 \;; \qquad \text{for } \frac{3\pi}{4} \le \omega \le \pi$$

From the equation, the digital filter specifications are:

$A_p = 0.9,\ \omega_p = \frac{\pi}{2}$ rad/sample

$A_s = 0.2,\ \omega_s = \frac{3\pi}{4}$ rad/sample

Sampling frequency $(F_s) = 1$ Hz.

$$T = \frac{1}{F_s} = \frac{1}{1\text{ Hz}} = 1 \text{ sec}$$

I. Equivalent analog filter using bilinear transformation is

$$\Omega_p = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right) = \frac{2}{1}\tan\left(\frac{\pi/2}{2}\right) = 2 \text{ rad/sec}$$

and,

$$\Omega_s = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right) = \frac{2}{1}\tan\left(\frac{3\pi/4}{2}\right) = 4.828427 \text{ rad/sec}$$

Required analog filter specifications are:

$A_p = 0.9,\qquad \Omega_p = 2$ rad/sec,

$A_s = 0.2,\qquad \Omega_s = 4.828427$ rad/sec

II. Calculating order of the filter:

$$N = \frac{\log\left[\dfrac{\frac{1}{A_s^2}-1}{\frac{1}{A_p^2}-1}\right]}{2\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$= \frac{\log\left(\frac{24}{0.234568}\right)}{2\log\left(\frac{4.828427}{2}\right)}$$

$$= 2.625483557$$

$$\therefore \; N \simeq 3$$

III. Calculating cut-off frequency since ω_c is not given:

$$\Omega_c = \frac{\Omega_p}{\left(\frac{1}{A_p^2} - 1\right)^{\frac{1}{2N}}}$$

$$= \frac{2}{\left(\frac{1}{0.9^2} - 1\right)^{\frac{1}{2\times 3}}}$$

$\therefore \ \Omega_c = 2.54674365$

IV. Determining poles of $H_a(s)$:

$p_i = \pm\Omega_c e^{j(N+2i+1)\pi/2N}$; $i = 0, 1, 2, \ldots\ldots., N-1$

For $i = 0$,

$$p_0 = \pm 2.54674365 e^{j(3+1)\pi/2N}$$

$$= \pm 2.54674365 e^{2\pi j/3}$$

$$= \pm 2.54674365\left[\cos\left(\frac{2\pi}{3}\right) + j\sin\left(\frac{2\pi}{3}\right)\right]$$

$$= \pm[(-1.27337) + j2.205544]$$

$\therefore \ p_0 = -1.27337 + j2.205544$ and $1.27337 - j2.205544$

For $i = 1$,

$$p_1 = \pm 2.54674365 e^{j(3+2+1)\pi/6}$$

$$= \pm 2.54674365 e^{j\pi}$$

$$= \pm 2.54674365[\cos(\pi) + j\sin(\pi)]$$

$$= \pm[(-2.54674365) + j \times 0]$$

$$= \pm(-2.54674365).$$

$\therefore \ p_1 = -2.54674365$ and 2.54674365

For $i = 2$,

$$p_2 = \pm 2.54674365 e^{j(3+4+1)\pi/6}$$

$$= \pm 2.54674365 e^{j8\pi/6}$$

$$= \pm 2.54674365 e^{4\pi j/3}$$

$$= \pm 2.54674365\left[\cos\left(\frac{4\pi}{3}\right) + j\sin\left(\frac{4\pi}{3}\right)\right]$$

$$= \pm[(-1.28237) + j(-2.2211)]$$

$\therefore \ p_2 = -1.28237 - j2.2211$ and $1.28237 + j2.2211$

Taking only the stable poles for filter design, we have

$p_0 = -1.27337 + j2.205544$

$p_1 = -2.54674365$

$p_2 = -1.28237 - j2.2211$

V. Calculating $H_a(s)$:

$$H_a(s) = \frac{\Omega_c^N}{(s - p_0)(s - p_1)(s - p_2)}$$

$$= \frac{2.54674365^3}{(s+1.27337 - j2.205544)(s+2.54674365)(s+1.28237+j2.2211)}$$

Here,

$p_0 \simeq -1.28 + j2.2$

$p_1 \simeq -2.55$

$p_2 \simeq -1.28 - j2.2$

So,

$$H_a(s) = \frac{16.58}{(s+1.28-j2.2)(s+2.55)(s+1.28+j2.2)}$$

$$= \frac{16.58}{\{(s+1.28)^2 - (j2.2)^2\}(s+2.55)}$$

$$= \frac{16.58}{(s^2+2.56s + 1.6384 - j^2 4.84)(s+2.55)}$$

$$= \frac{16.58}{(s+2.55)(s^2+2.56s+6.4784)}$$

VI. Calculating H(z) using bilinear transformation:

Put $s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = 2\frac{(z-1)}{(z+1)}$ in $H_a(s)$, then

$$H(z) = \frac{16.58}{\left(2\frac{(z-1)}{(z+1)}+2.55\right)\left[\left\{\frac{2(z-1)}{(z+1)}\right\}^2+2.56\times 2\left(\frac{z-1}{z+1}\right)+6.4784\right]}$$

$$= \frac{16.58}{\left(\frac{2z-2+2.55z+2.55}{z+1}\right)\left[\frac{4(z-1)^2}{(z+1)^2}+\frac{5.12(z-1)}{(z+1)}+6.4784\right]}$$

$$= \frac{16.58(z+1)^3}{(4.55z+0.55)(4z^2-8z+4+5.12z^2-5.12+6.4784z^2 12.9568z+6.4784)}$$

$$= \frac{16.58(z+1)^3}{(4.55z+0.55)(15.5984z^2+4.9568z+5.3584)}$$

which is the transfer function of the required digital filter using Butterworth approximation.

Example 6.5:

Design a LPF Butterworth filter using Impulse Invariance Method (IIM) with passband and stopband frequencies 200 Hz and 500 Hz respectively. The passband and stopband attenuations are 5 dB and 12 dB respectively. The sampling frequency is 5000 Hz.

[2078 Bhadra, 2072 Chaitra]

Solution:

Given,

Passband frequency $(F_p) = 200$ Hz

Stopband frequency $(F_s) = 500$ Hz

Passband attenuation $(A_p(dB)) = 5$ dB

Stopband attenuation $(A_s(dB)) = 12$ dB

Sampling frequency $(F_{sam}) = 5000$ Hz

Here, Time $(T) = \frac{1}{F_{sam}} = \frac{1}{5000} = 2 \times 10^{-4}$ sec

Now, $f_p = \frac{F_p}{F_{sam}} = \frac{200 \text{ Hz}}{5000 \text{ Hz}} = 0.04$ Hz

$\therefore\ \omega_p = 2\pi f_p = 2\pi \times 0.04 = 0.08\pi$ rad/sample

and, $f_s = \frac{F_s}{F_{sam}} = \frac{500 \text{ Hz}}{5000 \text{ Hz}} = 0.1$ Hz

$\therefore\ \omega_s = 2\pi f_s = 2\pi \times 0.1 = 0.2\pi$ rad/sample

I. Equivalent analog filter using impulse invariance method is,

$$\Omega_p = \frac{\omega_p}{T} = \frac{0.08\pi}{2 \times 10^{-4}} = 400\pi \text{ rad/sec}$$

$$\text{and, } \Omega_s = \frac{\omega_s}{T} = \frac{0.2\pi}{2 \times 10^{-4}} = 1000\pi \text{ rad/sec}$$

Required analog filter specifications are:

$A_p(dB) = 5$ dB, $\Omega_p = 400\pi$ rad/sec

$A_s(dB) = 12$ dB, $\Omega_s = 1000\pi$ rad/sec

II. Calculating order of the filter:

$$N = \frac{\log\left[\frac{10^{0.1A_s(dB)} - 1}{10^{0.1A_p(dB)} - 1}\right]}{2\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$= \frac{\log\left[\frac{10^{0.1\times12} - 1}{10^{0.1\times5} - 1}\right]}{2 \times \log\left(\frac{1000\pi}{400\pi}\right)}$$

$$= \frac{\log(6.867264181)}{2\log(2.5)}$$

$$= 1.051394352$$

$$\therefore\ N \simeq 2$$

III. Calculating cut-off frequency since ω_c is not given:

$$\Omega_c = \frac{\Omega_p}{(10^{0.1A_p(dB)} - 1)^{\frac{1}{2N}}}$$

$$= \frac{400\pi}{(10^{0.1\times5} - 1)^{\frac{1}{4}}}$$

$$\therefore\ \Omega_c = 1036.29165 \text{ rad/sec}$$

IV. Determining poles of $H_a(s)$:

$p_i = \pm\Omega_c e^{j(N+2i+1)\pi/2N}$; $i = 0, 1, 2, \ldots\ldots\ldots, N-1$

For $i = 0$,

$$p_0 = \pm 1036.29165 e^{j(2+1)\pi/4}$$

$$= \pm 1036.29165 e^{j3\pi/4}$$

$$= \pm 1036.29165\left[\cos\left(\frac{3\pi}{4}\right) + j\sin\left(\frac{3\pi}{4}\right)\right]$$

$$= \pm[(-732.7688) + j732.7688]$$

$\therefore\ p_0 = -732.7688+j732.7688$ and $732.7688-j732.7688$

For $i = 1$,

$$p_1 = \pm 1036.29165 e^{j(2+2+1)\pi/4}$$

$$= \pm 1036.29165 e^{j5\pi/4}$$

$$= \pm 1036.29165\left[\cos\left(\frac{5\pi}{4}\right) + j\sin\left(\frac{5\pi}{4}\right)\right]$$

$$= \pm[-732.7688 + j(-732.7688)]$$

$\therefore\ p_1 = -732.7688-j732.7688$ and $732.7688+j732.7688$

Taking only the stable poles for filter design, we have

$p_0 = -732.7688 + j732.7688$

$p_1 = -732.7688 - j732.7688$

V. Calculating $H_a(s)$:

$$H_a(s) = \frac{\Omega_c^N}{(s-p_0)(s-p_1)}$$

$$= \frac{1036.29165^2}{(s+732.7688-j732.7688)(s+732.7688+j732.7688)}$$

$$= \frac{1073.9 \times 10^3}{(s+732.7688)^2 - (j732.7688)^2}$$

$$\therefore\ H_a(s) = \frac{1073.9 \times 10^3}{(s+732.7688)^2 + (732.7688)^2}\ [\because j^2 = -1]$$

VI. Calculating H(z) using impulse invariance method:

We have,

$$H_a(s) = \frac{1073.9 \times 10^3}{(s+732.7688)^2 + (732.7688)^2}$$

$$= \frac{1073.9\times10^3}{732.7688} \times \frac{732.7688}{(s+732.7688)^2+(732.7688)^2}$$

$$= 1465.537 \times \frac{732.7688}{(s+732.7688)^2+(732.7688)^2}$$

Comparing with

$$\frac{b}{(s+a)^2+b^2} \rightarrow \frac{e^{-aT}\sin bTz^{-1}}{1-2e^{-aT}[\cos bT]z^{-1} + e^{-2aT}z^{-2}}$$

We have,

$$H(z) = \frac{1465.537\times e^{-732.7688\times2\times10^{-4}}\sin(732.7688\times2\times10^{-4})z^{-1}}{1-2e^{-732.7688\times2\times10^{-4}}\cos(732.7688\times2\times10^{-4})z^{-1} + e^{-2\times732.7688\times2\times10^{-4}}z^{-2}}$$

$$\therefore\ H(z) = \frac{184.83768z^{-1}}{1-1.7088z^{-1}+0.7459z^{-2}}$$

which is the transfer function of the required digital filter using Butterworth approximation.

Example 6.6:

Design a low pass digital IIR filter by bilinear transformation method to an approximate Butterworth low pass filter, if passband edge frequency is 0.26π radians and maximum deviation of 0.99 dB below 0 dB gain in the pass band. The maximum gain of –14.99 dB and frequency is 0.58π radians in the stop band. Consider sampling frequency of 0.5 Hz. [2080 Baishakh]

Solution:

Given,

Passband edge frequency (ω_p) = 0.26π radians

Passband attenuation (A_p(dB)) = 0.99 dB

Stopband edge frequency (ω_s) = 0.58π radians

Stopband attenuation (A_s(dB)) = 14.99 dB

Sampling frequency (F_{sam}) = 0.5 Hz

Note: *The –ve sign indicates attenuation*

Here, Time (T) $= \frac{1}{F_{sam}} = \frac{1}{0.5} = 2$ seconds

I. Equivalent analog filter using bilinear transformation is

$$\Omega_p = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right) = \frac{2}{2}\tan\left(\frac{0.26\pi}{2}\right)$$

$$= \tan(0.13\pi)$$

$$= 0.432738 \text{ rad/sec}$$

$$\Omega_s = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right) = \frac{2}{2}\tan\left(\frac{0.58\pi}{2}\right)$$

$$= \tan(0.29\pi)$$

$$= 1.289192 \text{ rad/sec}$$

Required analog filter specifications are

$A_p(dB) = 0.99$ dB, $\Omega_p = 0.432738$ rad/sec

$A_s(dB) = 14.99$ dB, $\Omega_s = 1.289192$ rad/sec

II. Calculating order of the filter:

$$N = \frac{\log\left[\frac{10^{0.1A_s(dB)}-1}{10^{0.1A_p(dB)}-1}\right]}{2\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$= \frac{\log\left[\frac{10^{0.1\times 14.99}-1}{10^{0.1\times 0.99}-1}\right]}{2\log\left(\frac{1.28919}{0.432738}\right)}$$

$$= \frac{\log(119.322152)}{2\log(2.9791467)}$$

$$= 2.19$$

$$\therefore \quad N \simeq 3$$

III. Calculating cut-off frequency since ω_c is not given:

$$\Omega_c = \frac{\Omega_p}{(10^{0.1A_p(dB)}-1)^{\frac{1}{2N}}}$$

$$= \frac{0.432738}{(10^{0.1\times 0.99}-1)^{\frac{1}{6}}}$$

$$\therefore \quad \Omega_c = 0.54305 \text{ rad/sec}$$

IV. Determining poles of $H_a(s)$:

$p_i = \pm\Omega_c e^{j(N+2i+1)\pi/2N}$; $i = 0, 1, 2, \dots\dots, N-1$

For i = 0,

$$p_0 = \pm 0.54305e^{j(3+1)\pi/6}$$

$$= \pm 0.54305e^{j2\pi/3}$$

$$= \pm 0.54305\left[\cos\left(\frac{2\pi}{3}\right) + j\sin\left(\frac{2\pi}{3}\right)\right]$$

$$= \pm[(-0.271525) + j0.470295]$$

$$\therefore \quad p_0 = -0.271525+j0.470295 \text{ and } 0.271525-j0.470295$$

For i = 1,

$$p_1 = \pm 0.54305e^{j(3+2+1)\pi/6}$$

$$= \pm 0.54305e^{j\pi}$$

$$= \pm 0.54305[\cos(\pi) + j\sin(\pi)]$$

$$= \pm 0.54305(-1 + j \times 0)$$

$$\therefore \quad p_1 = -0.54305 \text{ and } 0.54305$$

For i = 2,

$$p_2 = \pm 0.54305e^{j(3+4+1)\pi/6}$$

$$= \pm 0.54305e^{j4\pi/3}$$

$$= \pm 0.54305\left[\cos\left(\frac{4\pi}{3}\right) + j\sin\left(\frac{4\pi}{3}\right)\right]$$

$$= \pm[(-0.271525) + j(-0.470295)]$$

$$\therefore \quad p_2 = -0.27152 - j0.470295 \text{ and } 0.271525+j0.470295$$

Taking only the stable poles for filter design, we have

$p_0 = -0.27152 + j0.470295$

$p_1 = -0.54305$

$p_2 = -0.27152 - j0.470295$

V. Calculating $H_a(s)$:

$$H_a(s) = \frac{\Omega_c^{\ N}}{(s-p_0)(s-p_1)(s-p_2)}$$

$$= \frac{0.54305^3}{(s+0.27152-j0.470295)(s+0.54305)(s+0.27152+j0.470295)}$$

$$= \frac{0.160147}{[(s+0.27152)^2 - (j0.470295)^2](s+0.54305)}$$

$$= \frac{0.160147}{(s+0.54305)(s^2+0.54305s+0.0737258-j^2 0.221177)}$$

$$\therefore \quad H_a(s) = \frac{0.160147}{(s+0.54305)(s^2+0.54305s+0.2949)}$$

VI. Calculating H(z) using bilinear transformation:

Put $s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = \frac{2}{2}\left(\frac{z-1}{z+1}\right) = \left(\frac{z-1}{z+1}\right)$ in $H_a(s)$, then

$$H(z) = \frac{0.160147}{\left(\frac{z-1}{z+1}+0.54305\right)\left[\frac{(z-1)^2}{(z+1)^2}+0.54305\left(\frac{z-1}{z+1}\right)+0.2949\right]}$$

$$= \frac{0.160147}{\left(\frac{z-1+0.54305z+0.54305}{z+1}\right)\left[\frac{(z-1)^2+0.54305(z-1)(z+1)+0.2949(z+1)^2}{(z+1)^2}\right]}$$

$$= \frac{0.160147(z+1)^3}{(1.54305z-0.45695)(z^2-2z+1+0.54305z^2-0.54305z+0.2949z^2+0.5898z+0.2949)}$$

$$= \frac{0.160147(z+1)^3}{(1.54305z-0.45695)(1.83795z^2-1.4102z+0.75185)}$$

which is the transfer function of the required digital filter using Butterworth approximation.

Example 6.7:

Design a low pass discrete time Butterworth filter using bilinear transformation having following specifications:

Passband frequency (ω_p) = 0.25π radians

Stopband frequency (ω_s) = 0.55π radians

Passband ripple (δ_p) = 0.11

Stopband ripple (δ_s) = 0.21.

Consider sampling frequency of 0.5 Hz.

Also, convert the obtained digital low pass filter to high pass filter with new passband frequency, $\omega_p' = 0.45\pi$ using digital domain transformation.

[2079 Bhadra, 2071 Chaitra, 2069 Chaitra]

Solution:

Given, digital filter specifications are:

Passband frequency (ω_p) = 0.25π radians

Stopband frequency (ω_s) = 0.55π radians

Passband ripple (δ_p) = 0.11

Stopband ripple (δ_s) = 0.21

Sampling frequency (F_{sam}) = 0.5 Hz

Here, $T = \frac{1}{F_{sam}} = \frac{1}{0.5} = 2$ sec

Passband attenuation (A_p) = $1 - \delta_p = 1 - 0.11 = 0.89$

Stopband attenuation (A_s) = $\delta_s = 0.21$

I. Equivalent analog filter using bilinear transformation is:

$$\Omega_p = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right) = \frac{2}{2}\tan\left(\frac{0.25\pi}{2}\right) = 0.414235 \text{ rad/sec}$$

$$\Omega_s = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right) = \frac{2}{2}\tan\left(\frac{0.55\pi}{2}\right) = 1.1708495 \text{ rad/sec}$$

Required analog specifications are:

$A_p = 0.89$, $\Omega_p = 0.4142135$ rad/sec

$A_s = 0.21$, $\Omega_s = 1.1708495$ rad/sec

II. Calculating order of the filter:

$$N = \frac{\log\left[\frac{\frac{1}{A_s^2}-1}{\frac{1}{A_p^2}-1}\right]}{2\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$= \frac{\log(82.58466)}{2\log(2.82668)}$$

$$= 2.12386$$

$$\therefore \; N \simeq 3$$

III. Calculating cut-off frequency since ω_c is not given:

$$\Omega_c = \frac{\Omega_p}{\left(\frac{1}{A_p^2}-1\right)^{\frac{1}{2N}}}$$

$$= \frac{0.4142135}{\left(\frac{1}{0.89^2}-1\right)^{\frac{1}{6}}}$$

$$= 0.51766 \text{ rad/sec}$$

IV. Determining poles of $H_a(s)$:

$p_i = \pm\Omega_c e^{j(N+2i+1)\pi/2N}$; $i = 0, 1, 2, \ldots\ldots, N-1$

For i = 0,

$$p_0 = \pm 0.51766 e^{j(3+1)\pi/6}$$
$$= \pm 0.51766 e^{j2\pi/3}$$
$$= \pm 0.51766\left[\cos\left(\frac{2\pi}{3}\right) + j\sin\left(\frac{2\pi}{3}\right)\right]$$
$$= \pm(-0.25883 + j0.4483)$$

$\therefore \; p_0 = -0.25883 + j0.4483$ and $0.25883 - j0.4483$

For i = 1,

$$p_1 = \pm 0.51766 e^{j(3+2+1)\pi/6}$$
$$= \pm 0.51766 e^{j\pi}$$
$$= \pm 0.51766[\cos(\pi) + j\sin(\pi)]$$
$$= \pm 0.51766(-1 + j \times 0)$$

$\therefore \; p_1 = -0.51766$ and 0.51766

For i = 2,

$$p_2 = \pm 0.51766 e^{j(3+4+1)\pi/6}$$
$$= \pm 0.51766 e^{j4\pi/3}$$
$$= \pm 0.51766\left[\cos\left(\frac{4\pi}{3}\right) + j\sin\left(\frac{4\pi}{3}\right)\right]$$
$$= \pm[(-0.25883) + j(-0.4483)]$$

$\therefore \; p_2 = -0.25883 - j0.4483$ and $0.25883 + j0.4483$

Taking only the stable poles for filter design, we have

$p_0 = -0.25883 + j0.4483$

$p_1 = -0.51766$

$p_2 = -0.25883 - j0.4483$

V. Calculating $H_a(S)$:

$$H_a(s) = \frac{\Omega_c^N}{(s-p_0)(s-p_1)(s-p_2)}$$
$$= \frac{0.51766^3}{(s+0.25883-j0.4483)(s+0.51766)(s+0.25883+j0.4483)}$$

$$= \frac{0.138718}{[(s+0.25883)^2 - (j0.4483)^2](s+0.51766)}$$
$$= \frac{0.138718}{(s+0.51766)(s^2+0.51766s+0.066993-j^2 0.20097)}$$
$$\therefore \; H_a(s) = \frac{0.138718}{(s+0.51766)(s^2+0.51766s+0.267963)}$$

VI. Calculating H(z) using bilinear transformation:

Put $s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = \frac{2}{2}\left(\frac{z-1}{z+1}\right) = \frac{(z-1)}{(z+1)}$ in $H_a(s)$, then

$$H(z) = \frac{0.138718}{\left[\frac{(z-1)}{(z+1)} + 0.51766\right]\left[\frac{(z-1)^2}{(z+1)^2} + 0.51766\left(\frac{z-1}{z+1}\right) + 0.267963\right]}$$
$$= \frac{0.138718(z+1)^3}{(z-1+0.51766z+0.51766)(z^2-2z+1+0.51766z^2-0.51766+0.267963z^2+0.535926z+0.267963)}$$
$$\therefore \; H(z) = \frac{0.138718(z+1)^3}{(1.51766z-0.48234)(1.785623z^2-1.464074z+0.750303)}$$

which is the transfer function of the required low pass discrete time Butterworth filter obtained using bilinear transformation.

To convert this obtained digital low pass filter to high pass filter with new pass band frequency $\omega_p' = 0.45\pi$ using digital domain transformation, we have

Passband frequency of LPF, $\omega_p = 0.25\pi$ radians

Passband frequency of HPF, $\omega_p' = 0.45\pi$ radians

We put $z^{-1} = -\left[\frac{z^{-1}+\alpha}{1+\alpha z^{-1}}\right]$

where,

$$\alpha = -\frac{\cos[(\omega_p' + \omega_p)/2]}{\cos[(\omega_p' - \omega_p)/2]}$$
$$= -\frac{\cos[(0.45\pi + 0.25\pi)/2]}{\cos[(0.45\pi - 0.25\pi)/2]}$$
$$= -\frac{0.45399}{0.95105}$$
$$\therefore \; \alpha = -0.4773566$$

$$\text{So, } z^{-1} = -\left[\frac{z^{-1} - 0.4773566}{1 - 0.4773566z^{-1}}\right]$$

$$\text{or, } z = -\left[\frac{z - 0.4773566}{1 - 0.4773566z}\right]$$

We have,

$$H(z) = \frac{0.138718(z+1)^3}{(1.51766z-0.48234)(1.785623z^2-1.464074z+0.750303)}$$

It can also be written as

$$H(z) = \frac{0.138718(1+z^{-1})^3}{(1.51766-0.48234z^{-1})(1.78562-1.464074z^{-1}+0.750303z^{-2})}$$

Now, in HPF,

$$H(z) = \frac{0.138718\left[-\left(\frac{z-0.477}{1-0.477z}\right)+1\right]^3}{\left[1.52\times-\left(\frac{z-0.477}{1-0.477z}\right)-0.48\right]\left[1.78\times\left\{-\left(\frac{z-0.477}{1-0.477z}\right)\right\}^2-1.46\times-\left(\frac{z-0.477}{1-0.477z}\right)+0.75\right]}$$

$$\text{or, } H(z) = \frac{0.138718\times\frac{(-z+0.477+1-0.477z)^3}{(1-0.477z)^3}}{\left(\frac{-1.52z+0.725-0.48+0.22896z}{1-0.477z}\right)\left[1.78\times\frac{(z-0.477)^2}{(1-0.477z)^2}+1.46\times\left(\frac{z-0.477}{1-0.477z}\right)+0.75\right]}$$

$$= \frac{0.138718\times\frac{(-1.477z+1.477)^3}{(1-0.477z)^3}}{\left(\frac{-1.29104z+0.245}{1-0.477z}\right)\left[\frac{\begin{array}{c}1.78(z^2-0.954z+0.2275)+1.46(-0477z^2+1.2275z-0.477)+\\ 0.75(1-0.954z+0.2275z^2)\end{array}}{(1-0.477z)^2}\right]}$$

$$= \frac{0.138718(-1.477z+1.477)^3}{\begin{array}{c}(-1.29104z+0.245)\\ (1.78z^2-1.698z+0.405-0.696z^2+1.792z-0.696+0.75-0.7155z+0.17z^2)\end{array}}$$

$$= \frac{0.138718(1.477z-1.477)^3}{(1.29104z-0.245)(1.254z^2-0.6215z+0.459)}$$

$$\therefore\ H(z) = \frac{0.44696(z-1)^3}{(1.29104z-0.245)(1.254z^2-0.6215z+0.459)}$$

which is the required transfer function of equivalent high pass filter.

Example 6.8:

Design a digital low pass Butterworth filter by applying bilinear transformation techniques for the given specifications:

Passband peak to peak ripple ≤ 1 dB

Passband edge frequency = 1.2 kHz

Stopband attenuation ≥ 40 dB

Stopband edge frequency = 2.5 kHz

Sampling rate = 8 kHz

Solution:

Given digital filter specifications are:

Passband attenuation, $\alpha_p = 1$ dB

Passband edge frequency, $F_p = 1.2$ kHz

Stopband attenuation, $\alpha_s = 40$ dB

Stopband edge frequency, $F_s = 2.5$ kHz

Sampling rate, $F_{sam} = 8$ kHz

Here,

$$\text{Time (T)} = \frac{1}{F_{sam}} = \frac{1}{8\times10^3} = 1.25\times10^{-4}\ \text{sec}$$

Also, normalized digital filter specifications are:

$$f_p = \frac{F_p}{F_{sam}} = \frac{1.2\times10^3}{8\times10^3} = 0.15\ \text{Hz}$$

$$\therefore\ \omega_p = 2\pi f_p = 2\pi\times0.15 = 0.3\pi\ \text{rad/sample}$$

$$\text{and, } f_s = \frac{F_s}{F_{sam}} = \frac{2.5\times10^3}{8\times10^3} = 0.3125\ \text{Hz}$$

$$\therefore\ \omega_s = 2\pi f_s = 2\pi\times0.3125 = 0.625\pi\ \text{rad/sample}$$

I. Equivalent analog filter using bilinear transformation is

$$\Omega_p = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right) = \frac{2}{1.25\times10^{-4}}\tan\left(\frac{0.3\pi}{2}\right)$$

$$= 8152.407\ \text{rad/sec}$$

$$\Omega_s = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right) = \frac{2}{1.25\times10^{-4}}\tan\left(\frac{0.625\pi}{2}\right)$$

$$= 23945.69\ \text{rad/sec}$$

Required analog filter specifications are:

$\alpha_p = 1$ dB (A_p(dB)), $\Omega_p = 8152.407$ rad/sec

$\alpha_s = 40$ dB (A_s(dB)), $\Omega_s = 23945.69$ rad/sec

Now this question can be solved like previous questions.

Example 6.9:

Design a digital low-pass with the following specification:

i. Passband magnitude constant to 0.7 dB below the frequency of 0.15π.

ii. Stop-band attenuation at least 14 dB for the frequencies between 0.6π to π.

Use Butterworth approximation as a prototype and use bilinear transformation and IIM method to obtain the digital filter. [2075 Ashwin, 2074 Chaitra]

Solution:

Given digital filter specifications are:

Passband attenuation, $\alpha_p = 0.7$ dB

Passband edge frequency, $\omega_p = 0.15\pi$

Stopband attenuation; $\alpha_s = 14$ dB

Stopband edge frequency, $\omega_s = 0.6\pi$

Assume T = 1sec, we have,

I. Equivalent analog filter using bilinear transformation is

$$\Omega_p = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right) = \frac{2}{1}\tan\left(\frac{0.15\pi}{2}\right) = 0.48 \text{ rad/sec}$$

$$\Omega_s = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right) = \frac{2}{1}\tan\left(\frac{0.6\pi}{2}\right) = 2.7527 \text{ rad/sec}$$

Equivalent analog filter using impulse invariance method is:

$$\Omega_p = \frac{\omega_p}{T} = \frac{0.15\pi}{1} = 0.15\pi \text{ rad/sec} = 0.471 \text{ rad/sec}$$

$$\text{and, } \Omega_s = \frac{\omega_s}{T} = \frac{0.6\pi}{1} = 0.6\pi \text{ rad/sec} = 1.885 \text{ rad/sec}$$

From the required analog filter specification we can calculate other parameters using dB formula in this particular case. In this way, we can solve this problem like previous questions.

6.4 Properties of Chebyshev Filter, Properties of Elliptic Filter, Properties of Bessel Filter, Spectral Transformation

Chebyshev Approximation

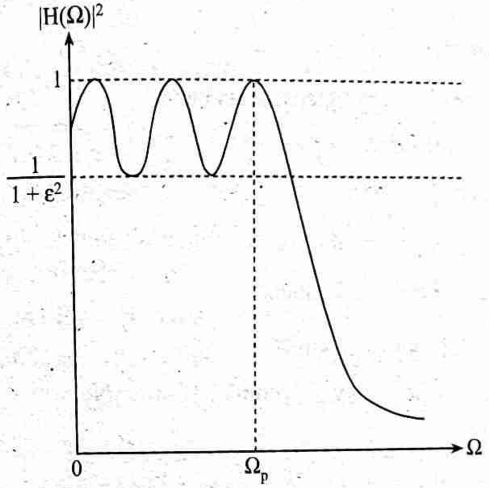


Fig.: Type-I Chebyshev filter characteristics

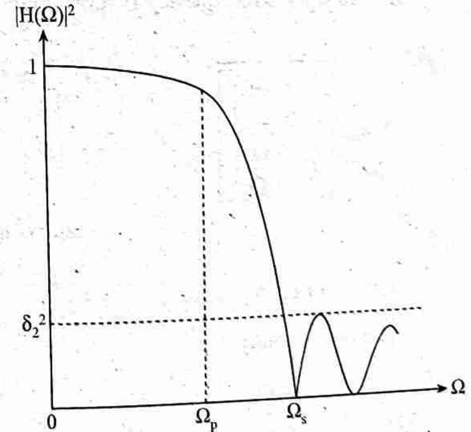


Fig.: Type-II Chebyshev filter characteristics

Type-I:

These filters are all pole filters. In the passband, these filters show equiripple behaviour and they have monotonic characteristics in the stopband.

The magnitude squared frequency response is given by

$$|H(\Omega)|^2 = \frac{1}{1+\varepsilon^2 C_N{}^2\left(\frac{\Omega_s}{\Omega_p}\right)}$$

where,

ε = ripple parameter in pass band

$= [10^{0.1A_p(dB)} - 1]^{1/2}$

$= \left[\frac{1}{A_p{}^2} - 1\right]^{1/2}$

Ω_p = Passband frequency

Ω_s = Stopband frequency

C_N = Chebyshev polynomial of order N

Type-II:

This filter consists of zeros as well as poles.

The magnitude squared frequency response is

$$|H(\Omega)|^2 = \frac{1}{1+\varepsilon^2\left[\frac{C_N{}^2\left(\frac{\Omega_s}{\Omega_p}\right)}{C_N{}^2\left(\frac{\Omega_s}{\Omega}\right)}\right]}$$

where,

ε = Ripple parameter in passband

Ω_p = Passband frequency

Ω_s = Stopband frequency

C_N = Chebyshev polynomial of order N

Note: *The major difference between Butterworth and Chebyshev filter is that the poles of Butterworth filter lies on the circle whereas the poles of Chebyshev filter lie on ellipse.*

- **Parameter ε**

  In dB, $\varepsilon = [10^{0.1A_p(dB)} - 1]^{1/2}$

  If A_p is not in dB, then

  $$\varepsilon = \left[\frac{1}{A_p{}^2} - 1\right]^{1/2}$$

- **Magnitude at cut-off frequency ($\Omega_c = 1$)**

  $$|H(\Omega)| = \frac{1}{\sqrt{1+\varepsilon^2}}$$

- **Order of the filter**

  When magnitude is expressed in dB, the order (N) of the filter is obtained by using

  $|H(j\Omega)|$ in dB $= -20\log_{10}\varepsilon - 6(N-1) - 20\log_{10}(\Omega_s)$

- **Poles of Chebyshev filter**

  $p_i = r\cos\theta_i + jR\sin\theta_i$

  where; $\theta_i = \frac{\pi}{2} + \frac{(2i+1)\pi}{2N}$; $i = 0, 1, 2, \ldots\ldots, N-1$

  $$r = \Omega_p\left(\frac{\beta^2-1}{2\beta}\right)$$

  $$R = \Omega_p\left(\frac{\beta^2+1}{2\beta}\right)$$

  and, $\beta = \left(\frac{\sqrt{1+\varepsilon^2}+1}{\varepsilon}\right)^{1/N}$

- **System transfer function**

  $$H_a(s) = \frac{i}{(s-p_0)(s-p_1)(s-p_2)\ldots..}$$

  where,

  $i = b_0$ for N being odd

  $= \frac{b_0}{\sqrt{1+\varepsilon^2}}$ for N being even

Example 6.10:

Design a digital low pass filter using Chebyshev filter that meets the following specifications passband magnitude characteristics that is constant within 1 dB for recurrences below $\omega = 0.2\pi$ and stopband attenuation of at least 15 dB for frequencies between $\omega = 0.3\pi$ and π. Use bilinear transformation.

Solution:

Given,

Passband ripple, $A_p(\text{dB}) = 1$ dB

Passband edge frequency, $\omega_p = 0.2\pi$ radians

Stopband attenuation, $A_s(\text{dB}) = 15$ dB

Stopband edge frequency, $\omega_s = 0.3\pi$ radians

Now,

Step 1: Calculation of analog filter's edge frequencies equivalent to given digital filter specifications using bilinear transformation:

$$\Omega_p = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right)$$

Assuming T = 1 sec,

$$\Omega_p = 2\tan\left(\frac{0.2\pi}{2}\right) = 0.64984 \text{ rad/sec}$$

$$\Omega_s = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right) = 2\tan\left(\frac{0.3\pi}{2}\right) = 1.01905 \text{ rad/sec}$$

Step 2: Calculation of ε:

$$\varepsilon = [10^{0.1A_p(\text{dB})} - 1]^{1/2} = [10^{0.1\times 1} - 1]^{1/2} = 0.508847$$

Step 3: Finding the order of filter:

$$|H(j\Omega) \text{ in dB} = -20\log_{10}\varepsilon - 6(N-1) - 20\log_{10}(\Omega_s)$$

$$\text{or, } -15 = -20\log_{10}(0.508847) - 6N + 6 - 20\log_{10}(1.01905).$$

$$\text{or, } -15 = 5.868255632 - 6N + 6 - 0.1639$$

$$\text{or, } -15 = 11.70435563 - 6N$$

$$\text{or, } 6N = 11.70435563 + 15$$

$$\text{or, } N = 4.45$$

$$\Rightarrow N \simeq 5$$

Step 4: Finding poles:

$$s_i = r\cos\theta_i + jR\sin\theta_i$$

$$\text{Here, } \beta = \left(\frac{\sqrt{1+\varepsilon^2}+1}{\varepsilon}\right)^{1/N}$$

$$= \left(\frac{\sqrt{1+(0.508847)^2}+1}{0.508847}\right)^{1/5}$$

$$= 1.343628$$

$$r = \Omega_p\left(\frac{\beta^2-1}{2\beta}\right) = 0.1947487$$

$$R = \Omega_p\left(\frac{\beta^2+1}{2\beta}\right) = 0.67839449$$

$$\theta_i = \frac{\pi}{2} + \frac{(2i+1)\pi}{2N}\ ;\ i = 0, 1, 2, 3, \ldots\ldots, N-1$$

$$\theta_0 = 1.88495 \left(\frac{6\pi}{10}\right)$$

$$\theta_1 = 2.51327 \left(\frac{8\pi}{10}\right)$$

$$\theta_2 = 3.14159\ (\pi)$$

$$\theta_3 = 3.76991 \left(\frac{12\pi}{10}\right)$$

$$\theta_4 = 4.39823 \left(\frac{14\pi}{10}\right)$$

So, $s_0 = -0.06 + j0.645$

$s_1 = -0.15755 + j0.3987$

$s_2 = -0.1947$

$s_3 = -0.15755 - j0.3987$

$s_4 = -0.06 - j0.645$

Step 5: Transfer function:

$$H_a(s) = \frac{i}{(s+0.06-j0.645)(s+0.15755-j0.3987)(s+0.1947)(s+0.15755+j0.3987)(s+0.06+j0.645)}$$

We know that, $i = b_0$ for N being odd

To calculate b_0,

$$b_0 = (-s_0 \times -s_1 \times -s_2 \times -s_3 \times -s_4) = 0.015$$

$$\therefore\ H_a(s) = \frac{0.015}{[(s+0.06)^2-(j0.645)^2]\,[(s+0.15755)^2-(j0.3987)^2]\,(s+0.1947)}$$

$$= \frac{0.015}{(s^2+0.12s+0.0036+0.416025)(s^2+0.315s+0.024822+0.15896)(s+0.1947)}$$

$$= \frac{0.015}{(s^2+0.12s+0.42)(s^2+0.315s+0.1838)(s+0.1947)}$$

Step 6: Calculating H(z) using bilinear transformation:

$$\text{Put } s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = 2\left(\frac{z-1}{z+1}\right)$$

$$\therefore\ H(z) = \frac{0.015}{\left[4\left(\frac{z-1}{z+1}\right)^2+0.12\left(\frac{z-1}{z+1}\right)+0.42\right]\left[4\left(\frac{z-1}{z+1}\right)^2+0.315\left(\frac{z-1}{z+1}\right)+0.1838\right]\left[\left(\frac{z-1}{z+1}\right)+0.1947\right]}$$

$$= \frac{0.015(z+1)^5}{[4(z-1)^2+0.12(z-1)(z+1)+0.42(z+1)^2]\,[4(z-1)^2+0.315(z-1)(z+1)+0.1838(z+1)^2]\,[z-1+0.1947(z+1)}$$

$$= \frac{0.015(z+1)^5}{(4z^2-8z+4+0.12z^2-0.12+0.42z^2+0.84z+0.42)(4z^2-8z+4+0.315z^2-0.315+0.1838z^2+0.3676z+0.1838)(z-1+0.1947z+0.1947)}$$

$$= \frac{0.015(z+1)^5}{(4.54z^2-7.16z+4.3)(4.4988z^2-7.6324z+3.8688)(1.1947z-0.8053)}$$

$$= \frac{0.015(z+1)^5}{17.5643(z^2-1.577z+0.947)(z^2-1.6965z+0.85996)(z-0.674)}$$

$$\therefore\ H(z) = \frac{8.54\times10^{-4}(z+1)^5}{(z^2-1.577z+0.947)(z^2-1.6965z+0.85996)(z-0.674)}$$

which is the transfer function of the required digital low pass Chebyshev filter obtained using bilinear transformation.

Spectral Transformation

Types of transformation	Transformation	Parameters
LPF	$z^{-1} \to \frac{z^{-1}-a}{1-az^{-1}}$	$a = \frac{\sin[(\omega_c-\omega_c')/2}{\sin[(\omega_c+\omega_c')/2}$
HPF	$z^{-1} \to -\frac{z^{-1}+a}{1+az^{-1}}$	$a = -\frac{\cos[(\omega_c+\omega_c')/2]}{\cos[(\omega_c-\omega_c')/2]}$
BPF	$z^{-1} \to -\frac{z^{-2}-a_1z^{-1}+a_2}{a_2z^{-2}-a_1z^{-1}+1}$	$a_1 = \frac{2\alpha k}{k+1}$; $a_2 = \frac{k-1}{k+1}$ where, $\alpha = \frac{\cos[(\omega_u+\omega_l)/2]}{\cos[(\omega_u-\omega_l)/2]}$ and, $k = \cot\left(\frac{\omega_u-\omega_l}{2}\right)\tan\left(\frac{\omega_c}{2}\right)$
BSF	$z^{-1} \to \frac{z^{-2}-a_1z^{-1}+a_2}{a_2z^{-2}-a_1z^{-1}+1}$	$a_1 = \frac{2\alpha}{k+1}$; $a_2 = \frac{1-k}{1+k}$ where, $\alpha = \frac{\cos[(\omega_u+\omega_l)/2]}{\cos[(\omega_u-\omega_l)/2]}$ and, $k = \tan\left(\frac{\omega_u-\omega_l}{2}\right)\tan\left(\frac{\omega_c}{2}\right)$

Chapter – 7

DISCRETE FOURIER TRANSFORM

7.1 Discrete Fourier Transform (DFT) Representation, Properties of DFT

Frequency analysis of discrete-time signals is usually and most conveniently performed on a digital signal processor, which may be a general-purpose digital computer or specially designed digital hardware. To perform frequency analysis on a discrete-time signal {x[n]}, we convert the time domain sequence to an equivalent frequency domain representation. This frequency domain representation is given by the Fourier transform $X(e^{j\omega})$ of the sequence {x[n]}. We know that $X(e^{j\omega})$ is a continuous function of frequency ω and therefore, it is not a computationally convenient representation of the sequence {x[n]}.

Thus, we consider the representation of a sequence {x[n]} by samples of its spectrum X(ω) or $X(e^{j\omega})$. Such a frequency-domain representation leads to the Discrete Fourier Transform (DFT), which is a powerful computational tool for performing frequency analysis of discrete-time signals.

Frequency Domain Sampling

Discrete Fourier Transform (DFT) is the equally spaced frequency samples of Discrete Time Fourier Transform (DTFT) over one period. Sampling is done at N equally spaced points over the period.

Here, we establish the relationship between the sampled Fourier transform and the DFT. The DFT is represented as X(k), which is the discrete frequency sequence of finite length used to represent discrete time sequence x[n] of finite length.

Let us consider an aperiodic finite duration discrete time signal x[n] with N terms whose DTFT is given by,

$$X(\omega) \text{ or } X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \text{ .... (i)}$$

Since $X(e^{j\omega})$ is periodic with period 2π, only samples in the fundamental frequency range are necessary. For convenience, we take N equidistant samples in the interval $0 \le \omega \le 2\pi$, i.e. 0, 1, 2, 3, ..., N – 1, with spacing $\frac{2\pi}{N}$.

If we evaluate equation (i) at $\omega = \frac{2\pi}{N}$, we obtain

$$X\left(e^{j\frac{2\pi}{N}}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\frac{2\pi}{N}n}$$

After sampling with N samples,

$$X\left(e^{j\frac{2\pi}{N}k}\right) = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} \text{ .... (ii)}$$

where k = 0, 1, 2, ..., N – 1 and $\omega = \frac{2\pi}{N}k$

In general, we can write

$$X(k) = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

Here, X(k) is the DFT of x[n]. Since summation is taken for N-points, it is also called *N-point DFT*.

We can obtain x[n] from X(k) which is called *Inverse Discrete Fourier Transform (IDFT).*

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N} = \text{IDFT}\{x(k)\}$$

Also, $X(k) = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$ is periodic with period N.

Proof:

$$X(k+N) = \sum_{n=0}^{N-1} x[n]e^{-j2\pi(k+N)n/N}$$

$$= \sum_{n=0}^{N-1} \{x[n]e^{-j2\pi kn/N} e^{-j2\pi n}\}$$

$$= \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \; [\because e^{-j2\pi n} = \cos(2\pi n) - j\sin(2\pi n) = 1]$$

$$\therefore X(k+N) = X(k)$$

Note:

Suppose $n = n + N$ in $x[n]$, then

$$x[n+N] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi k(n+N)/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N}\, e^{j2\pi k}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N} \; [\because e^{j2\pi k} = 1]$$

$$= x[n]$$

This shows x[n] to be periodic even though it is not. This may lead to confusion. Practically, x[n] is not periodic but theoretically it shows to be periodic. In order to mitigate this problem, we always take the range

$n = 0$ to $N - 1$

and , $k = 0$ to $N - 1$

Twiddle Factor

The DFT of x[n] is

$$X(k) = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

We can represent as $W_N = e^{-j2\pi/N}$ which is called *twiddle factor*. So,

$$X(k) = \sum_{n=0}^{N-1} x[n]W_N^{kn} \quad ; k = 0, 1, 2, \ldots\ldots, N-1$$

Similarly,

$$\text{IDFT: } x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn} \quad ; \quad n = 0, 1, 2, \ldots\ldots, N-1$$

The twiddle factor makes the computation of DFT a bit easy and fast.

Example 7.1:

Calculate DFT of x[n] = δ[n].

Solution:

Given, $x[n] = \delta[n]$

$\Rightarrow x[n] = 1$ for $n = 0$

0 otherwise

$$\text{Now, } X(k) = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

$$= x[0]e^{-j2\pi k\times 0/N}$$

$$= 1 \times 1$$

$$= 1 \text{ for all } k$$

For N point DFT, k = 0, 1, 2, ……, N – 1. So, X(0), X(1), X(2), ………, X(N – 1) has to be found.

Example 7.2:

Find the DFT of x[n] = δ(n – m).

Solution:

Given, $x[n] = 1$ for $n = m$

0 otherwise

$$\text{Here, } X(k) = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

$$= x[m]e^{-j2\pi km/N}$$

$$= 1 \times e^{-j2\pi km/N}$$

$$\therefore X(k) = W_N^{km}$$

Now,

$X(0) = 1$

$X(1) = e^{-j2\pi m/N} = W_N^{m}$

$X(2) = e^{-j2\pi .2m/N} = W_N^{2m}$

$$X(N-1) = e^{-j2\pi(N-1)m/N} = W_N^{(N-1)m}$$

Example 7.3:

Find the DFT of $x[n] = \cos\left(\frac{2\pi rn}{N}\right)$.

Solution:

Given, $x[n] = \cos\left(\frac{2\pi rn}{N}\right)$

$$= \frac{e^{j2\pi rn/N} + e^{-j2\pi rn/N}}{2}$$

$$= \frac{1}{2}(W_N^{-rn} + W_N^{rn})$$

Now, $X(k) = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$

$$= \sum_{n=0}^{N-1} x[n]W_N^{kn}$$

$$= \sum_{n=0}^{N-1} \left\{\frac{1}{2}(W_N^{-rn} + W_N^{rn})W_N^{kn}\right\}$$

$$= \frac{1}{2}\sum_{n=0}^{N-1} [W_N^{(k-r)n} + W_N^{(k+r)n}]$$

For k = r,

$$X(r) = \frac{1}{2}\sum_{n=0}^{N-1} [W_N^{(r-r)n} + W_N^{(r+r)n}]$$

$$= \frac{1}{2}\sum_{n=0}^{N-1} [1 + W_N^{2rn}]$$

$$= \frac{N}{2} + \frac{1}{2}\sum_{n=0}^{N-1} W_N^{2rn}]$$

For k = –r,

$$X(-r) = \frac{1}{2}\sum_{n=0}^{N-1} [W_N^{(-r-r)n} + W_N^{(-r+r)n}]$$

$$= \frac{1}{2}\sum_{n=0}^{N-1} [W_N^{-2rn} + 1]$$

$$= \frac{N}{2} + \frac{1}{2}\sum_{n=0}^{N-1} W_N^{-2rn}$$

For every other values of k beside r and –r, there will be some values of (k – r) and (k + r). For the sine and cosine graph plot, the sum of upper portion and lower portion of graph will always result in zero within the range 0 to N – 1.

Hence,

$$X(k) = \frac{N}{2} \text{ if } k = r \text{ or } k = -r \text{ or } k = N - r$$

$$= 0 \text{ otherwise}$$

The DFT as a Linear Transformation

The formulas for the DFT and IDFT can be expressed as:

$$X(k) = \sum_{n=0}^{N-1} x[n]W_N^{kn} \quad ; k = 0, 1, \ldots\ldots, N-1$$

$$\text{and, } x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn} \quad ; n = 0, 1, \ldots\ldots, N-1$$

where, by definition,

$W_N = e^{-j2\pi/N}$ which is an N^{th} *root of unity*.

The N-point DFT may be expressed in matrix form as

$$X_N = W_N^{kn}\, x_N$$

where

W_N^{kn} is matrix of the linear transformation.

The IDFT can be expressed in matrix form as

$$x_N = \frac{1}{N}W_N^{*} X_N$$

where

W_N^{*} = Conjugate of $W_N = e^{j2\pi/N}$

We observe that W_N is a symmetric matrix of order N × N that represents different values of twiddle factor.

DFT:

$$\begin{bmatrix} X(0)\\ X(1)\\ X(2)\\ X(3)\\ \vdots\\ X(N-1) \end{bmatrix} = \begin{array}{cc} & \begin{array}{cccccc} n=0 & n=1 & n=2 & n=3 & \cdots\cdots & n=N-1 \end{array} \\ \begin{array}{c} k=0\\ k=1\\ k=2\\ k=3\\ \vdots\\ k=N-1 \end{array} & \begin{bmatrix} W_N^{kn} & W_N^{kn} & W_N^{kn} & W_N^{kn} & \cdots\cdots & W_N^{kn}\\ W_N^{kn} & W_N^{kn} & W_N^{kn} & W_N^{kn} & \cdots\cdots & W_N^{kn}\\ W_N^{kn} & W_N^{kn} & W_N^{kn} & W_N^{kn} & \cdots\cdots & W_N^{kn}\\ W_N^{kn} & W_N^{kn} & W_N^{kn} & W_N^{kn} & \cdots\cdots & W_N^{kn}\\ \vdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots\\ W_N^{kn} & W_N^{kn} & W_N^{kn} & W_N^{kn} & \cdots\cdots & W_N^{kn} \end{bmatrix} \end{array} \begin{bmatrix} x(0)\\ x(1)\\ x(2)\\ x(3)\\ \vdots\\ x(N-1) \end{bmatrix}$$

IDFT:

$$\begin{bmatrix} x(0)\\ x(1)\\ x(2)\\ x(3)\\ \vdots\\ x(N-1) \end{bmatrix} = \frac{1}{N} \begin{array}{cc} & \begin{array}{cccccc} n=0 & n=1 & n=2 & n=3 & \cdots\cdots & n=N-1 \end{array} \\ \begin{array}{c} k=0\\ k=1\\ k=2\\ k=3\\ \vdots\\ k=N-1 \end{array} & \begin{bmatrix} W_N^{*kn} & W_N^{*kn} & W_N^{*kn} & W_N^{*kn} & \cdots\cdots & W_N^{*kn}\\ W_N^{*kn} & W_N^{*kn} & W_N^{*kn} & W_N^{*kn} & \cdots\cdots & W_N^{*kn}\\ W_N^{*kn} & W_N^{*kn} & W_N^{*kn} & W_N^{*kn} & \cdots\cdots & W_N^{*kn}\\ W_N^{*kn} & W_N^{*kn} & W_N^{*kn} & W_N^{*kn} & \cdots\cdots & W_N^{*kn}\\ \vdots & \vdots & \vdots & \vdots & \cdots\cdots & \vdots\\ W_N^{*kn} & W_N^{*kn} & W_N^{*kn} & W_N^{*kn} & \cdots\cdots & W_N^{*kn} \end{bmatrix} \end{array} \begin{bmatrix} X(0)\\ X(1)\\ X(2)\\ X(3)\\ \vdots\\ X(N-1) \end{bmatrix}$$

Example 7.4:

Calculate 2-point DFT of x[n] = {1,1}.

Solution:

For 2-point DFT, N = 2

So, n = 0,1

k = 0,1

Here, $X(k) = \sum_{n=0}^{N-1} x[n]W_N^{kn}$; for k = 0, 1, ..., N – 1

Using 2 × 2 transformation matrix,

$$\begin{bmatrix} X(0)\\ X(1) \end{bmatrix} = \begin{bmatrix} W_2^0 & W_2^0\\ W_2^0 & W_2^1 \end{bmatrix} \begin{bmatrix} x(0)\\ x(1) \end{bmatrix}$$

We know,

$W_N^{kn} = e^{-j2\pi kn/N}$

So, $W_2^0 = e^{-j2\pi\times 0/2} = e^0 = 1$

$W_2^1 = e^{-j2\pi\times 1/2} = e^{-j\pi} = \cos(\pi) - j\sin(\pi) = -1 - 0 = -1$

Hence,

$$\begin{bmatrix} X(0)\\ X(1) \end{bmatrix} = \begin{bmatrix} 1 & 1\\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1\\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1\\ 1-1 \end{bmatrix} = \begin{bmatrix} 2\\ 0 \end{bmatrix}$$

$\therefore$ X(0) = 2, X(1) = 0

Example 7.5:

Calculate 4-point DFT of x[n] = {1, 1, 0, 0}.

Solution:

For 4-point DFT, N = 4

So, n = 0, 1, 2, 3

k = 0, 1, 2, 3

Using 4 × 4 transformation matrix, we get

$$\begin{bmatrix} X(0)\\ X(1)\\ X(2)\\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0\\ W_4^0 & W_4^1 & W_4^2 & W_4^3\\ W_4^0 & W_4^2 & W_4^4 & W_4^6\\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0)\\ x(1)\\ x(2)\\ x(3) \end{bmatrix}$$

We know,

$W_N^{kn} = e^{-j2\pi kn/N}$

So, $W_4^0 = e^{-j2\pi\times 0/4} = e^0 = 1$

$W_4^1 = e^{-j2\pi\times 1/4} = e^{-j\pi/2} = \cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right) = 0 - j \times 1 = -j$

$W_4^2 = e^{-j2\pi\times 2/4} = e^{-j\pi} = \cos(\pi) - j\sin(\pi) = -1$

$W_4^3 = e^{-j2\pi\times 3/4} = e^{-j6\pi/4} = \cos\left(\frac{6\pi}{4}\right) - j\sin\left(\frac{6\pi}{4}\right) = 0 - j(-1) = j$

$W_4^4 = e^{-j2\pi\times 4/4} = e^{-j2\pi} = \cos(2\pi) - j\sin(2\pi) = 1 - j \times 0 = 1$

$$W_4^6 = e^{-j2\pi\times6/4} = e^{-j3\pi} = \cos(3\pi) - j\sin(3\pi) = -1 - j\times0 = -1$$

$$W_4^9 = e^{-j2\pi\times9/4} = e^{-j9\pi/2} = \cos\left(\frac{9\pi}{2}\right) - j\sin\left(\frac{9\pi}{2}\right) = 0 - j(1) = -j$$

Hence,

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+0+0 \\ 1-j-0+0 \\ 1-1+0-0 \\ 1+j-0-0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

Hence,

$X(0) = 2, X(1) = 1 - j, X(2) = 0, X(3) = 1 + j$

Properties of DFT

DFT is a set of N samples {X(k)} of the Fourier transform $X(e^{j\omega})$ for a finite-duration sequence {x[n]} of length $L \le N$. The sampling of $X(e^{j\omega})$ occurs at the N equally spaced frequencies $\omega = 2\pi k/N$; k = 0, 1, 2, ..., N − 1. It was demonstrated that the N samples {X(k)} uniquely represent the sequence {x(n)} in the frequency domain. The important properties of the DFT are mentioned below. They resemble the properties of Fourier series, Fourier transform, and z-transform. A good understanding of these properties is extremely helpful in the application of the DFT to practical problems.

I. Periodicity:

If x[n] and X(k) are an N-point DFT pair, then

$x[n + N] = x[n]$ for all n

$X(k + N) = X(k)$ for all k

That is,

$$x[n] \xleftrightarrow[N]{\text{DFT}} X(k)$$

where, $X(k) = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}$ is periodic with period N.

Proof:

For k = k + N, we have

$$X(k+N) = \sum_{n=0}^{N-1} x[n]e^{-j2\pi(k+N)n/N} \; [\because W_N = e^{-j2\pi/N}]$$

$$= \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \times e^{-j2\pi n}$$

$$= \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \times 1 \; [\because e^{-j2\pi n} = 1]$$

$$= \sum_{n=0}^{N-1} x[n]W_N^{kn}$$

$\therefore\ X(k + N) = X(k)$

II. Linearity:

If $x_1[n] \xleftrightarrow[N]{\text{DFT}} X_1(k)$

and, $x_2[n] \xleftrightarrow[N]{\text{DFT}} X_2(k)$

Then, $a_1x_1[n] + a_2x_2[n] \xleftrightarrow[N]{\text{DFT}} a_1X_1(k) + a_2X_2(k)$

Proof:

We have,

$$X(k) = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} [a_1x_1[n] + a_2x_2[n]]e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} a_1x_1[n]e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} a_2x_2[n]e^{-j2\pi kn/N}$$

$$= a_1 \sum_{n=0}^{N-1} x_1[n]e^{-j2\pi kn/N} + a_2 \sum_{n=0}^{N-1} x_2[n]e^{-j2\pi kn/N}$$

$$= a_1X_1(k) + a_2X_2(k)$$

III. Symmetry:

If x[n] and X(k) are complex valued sequence

i.e. $x[n] = x_R[n] + jx_I[n]\ ;\ 0 \le n \le N-1$

$X(k) = X_R(k) + jX_I(k)\ ;\ 0 \le k \le N-1$

a. If x[n] is real valued

$X(N-k) = X^*(k) = X(-k)$

b. If x[n] is real and even

$x[n] = x[N-n]\ ;\ 0 \le n \le N-1$

c. If x[n] is real and odd

$x[n] = -x[N-n]\ ;\ 0 \le n \le N-1$

d. If x[n] is purely imaginary

$x[n] = jx_I[n]$

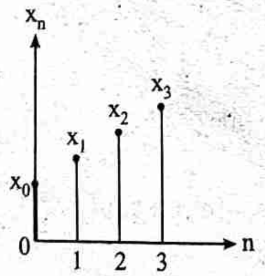
VI. Circular shift:

In DFT operation, going beyond the range of 0 to N – 1 for k and n may result in unexpected results.

Suppose $x[n] = \{\underset{\uparrow}{x_0}, x_1, x_2, x_3\}$

Here, N = 4 and x[n] starts from n = 0 ending at n = 3.

Let $x_0 < x_1 < x_2 < x_3$ then the graphical representation of x[n] will be



If we perform normal shifting, then x[n + 1] or x[n – 1] gives undesirable value in the case of DFT since they move out of range. So in DFT, to perform the function of shifting, we need circular shift.

In circular shift, the range is always maintained between 0 and N–1. If x[n] is the original signal, $x[n-a]_N$ and $x[n+a]_N$ are circular shift of the original signal x[n] where N is the number of elements in x[n], then

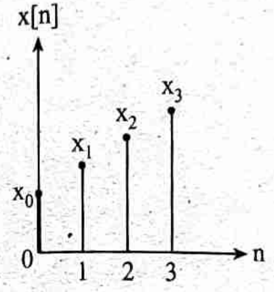


Fig.: Linear scale

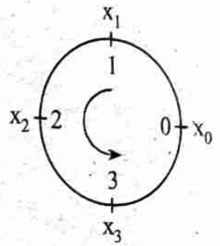
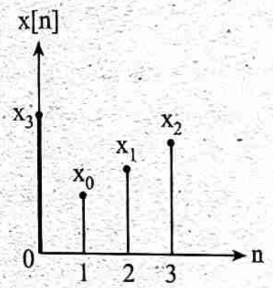
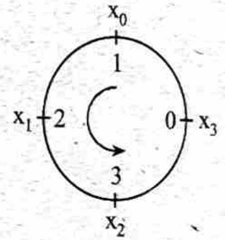


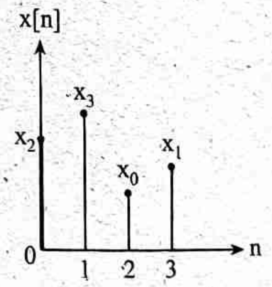
Fig.: Circular scale

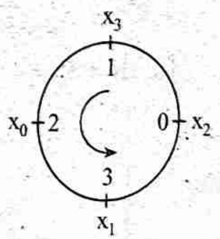
i. $x[n-1]_4$



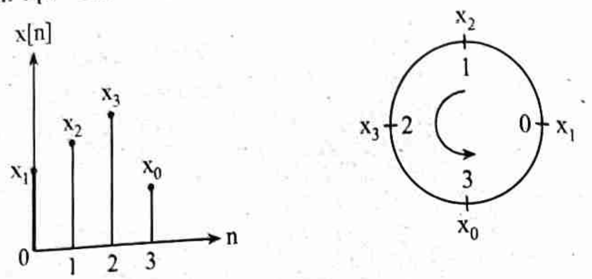


ii. $x[n-2]_4$

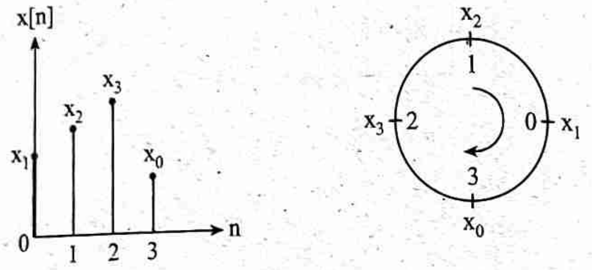




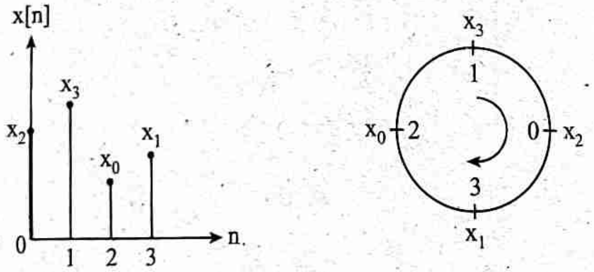
iii. $x[n-3]_4$



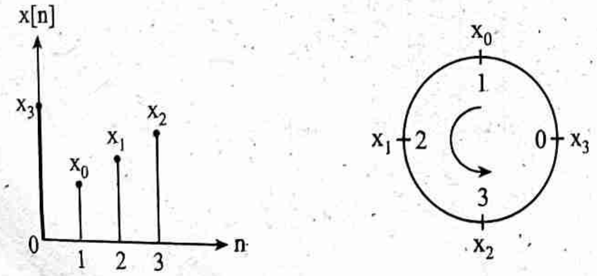
iv. $x[n+1]_4$



v. $x[n+2]_4$



vi. $x[n+3]_4$



V. Time shift and frequency shift:

i. Time shift property:

If $x[n] \xleftrightarrow[N]{DFT} X(k)$, then

$$x[n-n_0]_N \xleftrightarrow[N]{DFT} W_N^{kn_0} X(k)$$

where, $W_N^{kn_0} = e^{-j2\pi kn_0/N}$

Shifting the sequence in time domain by n_0 samples is equivalent to multiplying the sequence in frequency domain by $W_N^{kn_0}$ or $e^{-j2\pi kn_0/N}$.

ii. Frequency shift property:

If $x[n] \xleftrightarrow[N]{DFT} X(k)$, then

$$x[n]W_N^{-k_0 n} \xleftrightarrow[N]{DFT} X(k-k_0)_N$$

or, $x[n]W_N^{k_0 n} \xleftrightarrow[N]{DFT} X(k+k_0)_N$

Multiplication of sequence x[n] with the complex exponential sequence $e^{j2\pi k_0 n/N}$ is equivalent to the circular shift of the DFT by k_0 units in frequency.

VI. Time reversal property:

If $x[n] \xleftrightarrow[N]{DFT} X(k)$, then

$$x[-n]_N = \xleftrightarrow[N]{DFT} X(-k)_N$$

Since the range of n and k must be between 0 and N – 1, we write

$x[-n]_N = x[N-n]$

and, $X(-k)_N = X(N-k)$

Proof:

$$DFT\{x[-n]_N\} = DFT\{x[N-n]\}$$

$$= \sum_{n=0}^{N-1} x[n]e^{-j2\pi k(N-n)/N}$$

$$= \sum_{n=0}^{N-1} x[n]e^{-j2\pi k} \times e^{-j2\pi k(-n)/N}$$

$$= \sum_{n=0}^{N-1} x[n]e^{-j2\pi(-k)n/N} \times 1 \qquad [\because e^{-j2\pi k} = 1]$$

$$= X(-k)$$

$$= X(N-k)$$

VII. Circular correlation:

If $x[n] \xleftrightarrow[N]{DFT} X(k)$

and, $y[n] \xleftrightarrow[N]{DFT} Y(k)$, then

$$\tilde{r}_{xy}(l) \xleftrightarrow[N]{DFT} \tilde{R}_{xy}(k) = X(k)Y^*(k)$$

where, $\tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x[n]y^*(n-l)_N$

$$= \sum_{n=0}^{N-1} x[n]y^*[-(l-n)]_N$$

$$= x(l) \, Ⓝ \, y^*(-l)$$

$$\therefore \; \tilde{R}_{xy}(k) = DFT(\tilde{r}_{xy}(l)) = X(k)Y^*(k)$$

VIII. Parseval's theorem:

If $x[n] \xleftrightarrow[N]{DFT} X(k)$, then

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

Proof:

$$\sum_{n=0}^{N-1} |x(n)|^2 = \sum_{n=0}^{N-1} x[n]x^*[n] \text{ .... (i)}$$

Applying,

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N}$$

$$x^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)e^{-j2\pi kn/N}$$

Replacing x*[n] in equation (i), we get

$$\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{n=0}^{N-1} x[n]x^*[n]$$

$$= \sum_{n=0}^{N-1} x[n] \times \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)e^{-j2\pi kn/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X^*(k) \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)X(k)$$

$$\therefore \; \sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

IX. Circular convolution:

Convolution in time domain results in the multiplication in the frequency domain.

Let us assume, $x_1(n)$ is defined from 0 to N – 1 i.e. length of $x_1(n)$ is N. Also, $x_2(n)$ is defined from 0 to N – 1 i.e. length of $x_2(n)$ is N.

Here, the linear convolution is given by:

$$y(n) = x_1(n) * x_2(n)$$

$$= \sum_{m=0}^{N-1} x_1(m)\, x_2(n-m)$$

The length of y(n) is 2N – 1 which is way beyond N – 1 and hence is not suitable for DFT operation. So, in order to mitigate this issue, we define circular convolution.

The multiplication of two DFTs is equivalent to the circular convolution of their sequences in time domain.

Mathematically,

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$

and, $x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$, then

$$x_1(n) \, Ⓝ \, x_2(n) \xleftrightarrow[N]{DFT} X_1(k)\, X_2(k)$$

where,

$x_1(n)$ $\textcircled{N}$ $x_2(n)$ denotes the circular convolution of the sequence $x_1(n)$ and $x_2(n)$.

Let us assume,

$x_1(n)$ is defined from 0 to N – 1 i.e. length of $x_1(n)$ is N and $x_2(n)$ is defined from 0 to N – 1 i.e. length of $x_2(n)$ is N then circular convolution of $x_1(n)$ and $x_2(n)$ is defined as

$$y(n) = x_1(n) \textcircled{N} x_2(n)$$

$$= \sum_{m=0}^{N-1} x_1(m)\, x_2(n-m)_N$$

Here, the length of y(n) is N.

Example 7.6:

Find circular convolution:

$x_1[n] = \{1, 2, 0, 1\}$ and
↑

$x_2[n] = \{2, 2, 1, 1\}$
↑

Solution:

N = 4

$$y[n] = x_1[n] \textcircled{N} x_2[n]$$

$$= \sum_{m=0}^{N-1} x_1[m]x_2[n-m]_N$$

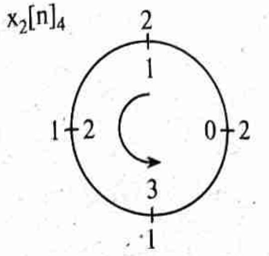
$$= \sum_{m=0}^{3} x_1[m]x_2[n-m]_4$$

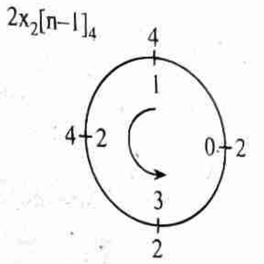
$$= x_1(0)x_2(n)_4+x_1(1)x_2(n-1)_4+x_1(2)x_2(n-2)_4+x_1(3)x_2(n-3)_4$$

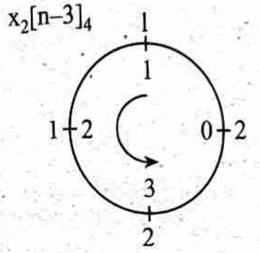
$$= 1 \times x_2(n)_4 + 2 \times x_2(n-1)_4 + 0 \times x_2(n-2)_4 + 1 \times x_2(n-3)_4$$

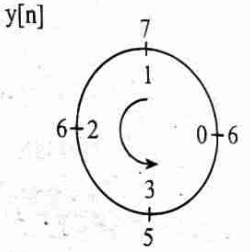
$$\therefore\ y[n] = x_2(n)_4 + 2x_2(n-1)_4 + x_2(n-3)_4$$

Now, plotting each elements in circular scale and performing circular shift, we have









$\therefore\ y[n] = \{6, 7, 6, 5\}$
↑

Linear Convolution by Circular Convolution

In linear convolution, we have

$$y[n] = x[n] * h[n]$$

$$= \sum_{m=0}^{N-1} x[m]h[n-m]$$

In circular convolution, we have

$$z[n] = x[n] \textcircled{N} h[n]$$

$$= \sum_{m=0}^{N-1} x[m]h[n-m]_N$$

Here,

Length of x[n] = N

Length of h[n] = N

Length of y[n] = 2N – 1

Length of z[n] = N

So, linear convolution can be performed using circular convolution by doing sufficient zero padding. *Zero padding* is a technique of extending a discrete signal by appending zeros to its end. It doesn't alter the original signal's content but increases its length with additional zero-valued samples.

Example 7.7:

Find linear convolution by circular convolution:

$x[n] = \{-1, 1\}$, $h[n] = \{2, 3, 1, -2\}$

$\uparrow$ $\uparrow$

Solution:

Here, Length of $x[n] = 2$

Length of $h[n] = 4$

Suppose $y[n]$ is the linear convolution, then the length of $y[n]$ is $2 + 4 - 1 = 5$.

Now, performing zero padding, we get

$x[n] = \{-1, 1, 0, 0, 0\}$

$\uparrow$

$h[n] = \{2, 3, 1, -2, 0\}$

$\uparrow$

Using circular convolution, we have

$$y[n] = \sum_{m=0}^{N-1} x[m]\, h[n-m]_N$$

$$= \sum_{m=0}^{4} x[m]\, h[n-m]_N$$

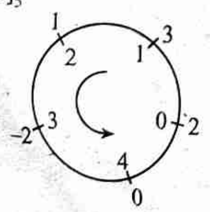
$$= x[0]h[n]_5 + x[1]h[n-1]_5 + x[2]h[n-2]_5 + x[3]h[n-3]_5 + x[4]h[n-4]_5$$

$$= -1 \times h[n]_5 + 1 \times h[n-1]_5 + 0 + 0 + 0$$

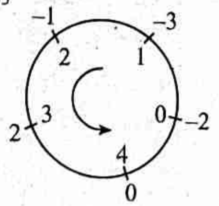
$$\therefore\ y[n] = -h[n]_5 + h[n-1]_5$$

Plotting each elements in circular scale, we have

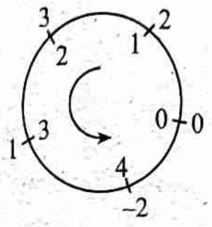
$h[n]_5$



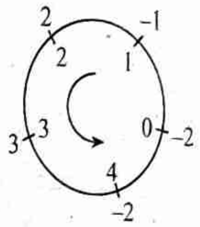
$-h[n]_5$



$h[n-1]_5$



$y[n]$



$\therefore\ y[n] = \{-2, -1, 2, 3, -2\}$

$\uparrow$

Difference between DTFT and DFT

Discrete-Time Fourier Transform (DTFT)	Discrete Fourier Transform (DFT)
i. The input signal is an infinite length discrete time signal $x[n]$, where $n \in Z$.	i. The input signal is a finite length discrete time signal of length N; $x[n]$ where $n = 0, 1, \ldots\ldots, N-1$
ii. Output of DTFT is continuous frequency response $X(e^{j\omega})$, $\omega \in [-\pi, \pi]$. Mathematically, $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$	ii. Output of DFT is discrete frequency components $X(k)$, where $k = 0, 1, \ldots\ldots, N-1$. Mathematically, $X(k) = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$
iii. The output is periodic with period 2π i.e. $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$	iii. The output is periodic with period N i.e. $X(k + N) = X(k)$
iv. It is used for theoretical analysis of discrete-time signals.	iv. It is used in practical digital signal processing (DSP) applications.

Example 7.8:

Obtain the circular convolution of the following sequences:

$x_1[n] = \{1, 2, 3, 1\}$ and

$x_2[n] = \{4, 3, 2, 2\}$ **[2080 Bhadra, 2080 Baishakh]**

Solution:

Given,

$x_1[n] = \{1, 2, 3, 1\}$

$x_2[n] = \{4, 3, 2, 2\}$

Here, N = 4

Using circular convolution, we get

$y[n] = x_1[n] \textcircled{N} x_2[n]$

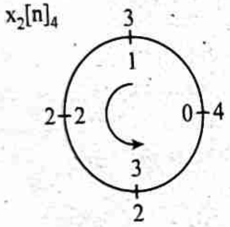
$$= \sum_{m=0}^{N-1} x_1[m]x_2[n-m]_N$$

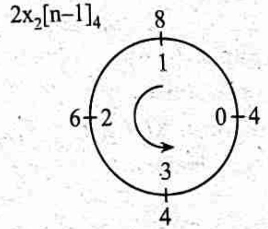
$$= \sum_{m=0}^{3} x_1[m]x_2[n-m]_4$$

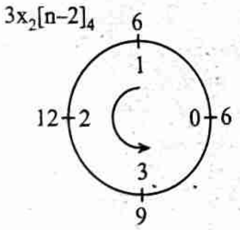
$$=x_1[0]x_2[n]_4+x_1[1]x_2[n-1]_4+x_1[2]x_2[n-2]_4+x_1[3]x_2[n-3]_4$$

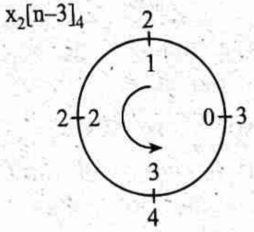
$$= 1 \times x_2[n]_4 + 2 \times x_2[n-1]_4 + 3 \times x_2[n-2]_4 + 1 \times x_2[n-3]_4$$

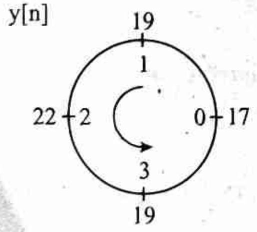
Now, plotting each elements in circular scale, we get











$\therefore\ y[n] = \{\underset{\uparrow}{17}, 19, 22, 19\}$

Example 7.9:

If $X_1(k)$ and $X_2(k)$ are DFT of sequences $x_1[n] = \{1, 0, 0, 1\}$ and $x_2[n] = \{2, 0, 2\}$ respectively, then find the sequence $x_3[n]$; if DFT of $x_3[n]$ is given by $X_3(k) = X_1(k)\ X_2(k)$. **[2079 Bhadra]**

Solution:

Given,

$x_1[n] = \{1, 0, 0, 1\}$

$x_2[n] = \{2, 0, 2\}$

and, $X_3(k) = X_1(k)\ X_2(k)$

Now, using convolution property of DFT, we know

$$x_1[n] \textcircled{N} x_2[n] \xleftrightarrow[N]{DFT} X_1(k)\ X_2(k)$$

Since $X_1(k)$ and $X_2(k)$ are DFT of $x_1[n]$ and $x_2[n]$ respectively and DFT of $x_3[n]$ is given by $X_3(k)$.

So, $x_3[n] = x_1[n] \textcircled{N} x_2[n]$

or, $x_3[n] = \sum_{m=0}^{N-1} x_1[m]x_2[n-m]_N$

Performing necessary zero padding,

$x_1[n] = \{1, 0, 0, 1\}$

$x_2[n] = \{2, 0, 2, 0\}$; N = 4

Therefore,

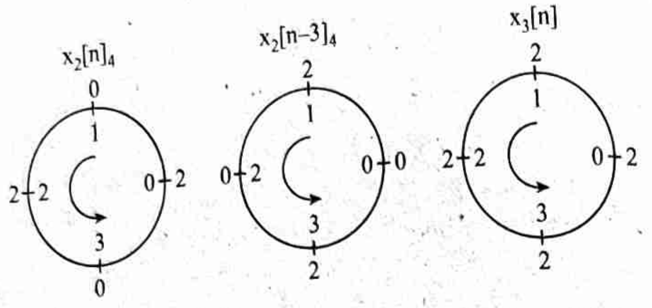
$$x_3[n] = \sum_{m=0}^{3} x_1[m]\ x_2[n-m]_4$$

$$= x_1[0]x_2[n]_4+x_1[1]x_2[n-1]_4+x_1[2]x_2[n-2]_4+x_1[3]x_2[n-3]_4$$

$$= 1 \times x_2[n]_4 + 0 \times x_2[n-1]_4 + 0 \times x_2[n-2]_4 + 1 \times x_2[n-3]_4$$

$\therefore\ x_3[n] = x_2[n]_4 + x_2[n-3]_4$

Plotting each elements in circular scale, we get



$\therefore\ x_3[n] = \{2, 2, 2, 2\}$
$\quad\quad\quad\quad\quad\uparrow$

Example 7.10:

Find the circular convolution of the sequence $x_1[n] = \{1, -1, -2, 3, -1\}$ and $x_2[n] = \{1, 2, 3\}$.

[2079 Baishakh]

Solution:

Given,

$x_1[n] = \{1, -1, -2, 3, -1\}$

$x_2[n] = \{1, 2, 3\}$

For 5-point DFT, performing necessary zero padding, we have

$x_2[n] = \{1, 2, 3, 0, 0\}$

Here, N = 5

Using circular convolution property of DFT, we get

$$y[n] = x_1[n] \text{Ⓝ} x_2[n]$$

$$= x_2[n] \text{Ⓝ} x_1[n]$$

$$= \sum_{m=0}^{N-1} x_2[m]x_1[n-m]_N$$

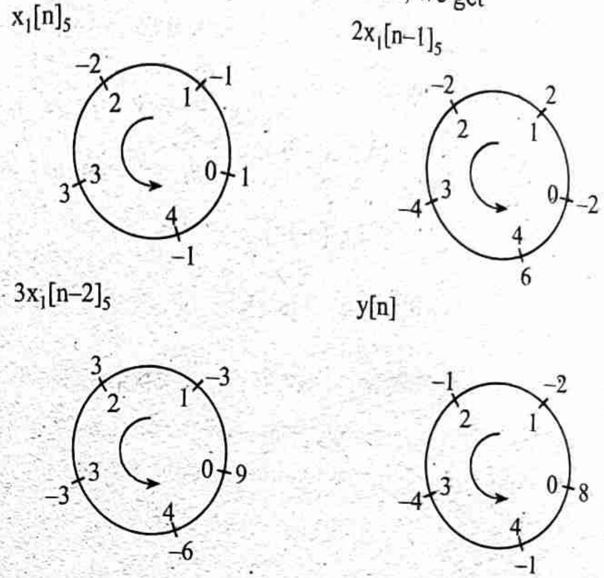
$$= \sum_{m=0}^{4} x_2[m]x_1[n-m]_5$$

$$= x_2[0]x_1[n]_5 + x_2[1]_5x_1[n-1]_5 + x_2[2]x_1[n-2]_5 + x_2[3]x_1[n-3]_5 + x_2[4]x_1[n-4]_5$$

$$= 1 \times x_1[n]_5 + 2 \times x_1[n-1]_5 + 3 \times x_1[n-2]_5 + 0 + 0$$

$$\therefore\ y[n] = x_1[n]_5 + 2x_1[n-1]_5 + 3x_1[n-2]_5$$

Plotting each element in circular scale, we get



$\therefore\ y[n] = \{8, -2, -1, -4, -1\}$
$\quad\quad\quad\quad\ \uparrow$

Example 7.11:

Find $x_3[n]$ if DFT of $x_3[n]$ is given by $X_3(k) = X_1(k)\ X_2(k)$ where $X_1(k)$ and $X_2(k)$ are 4-point DFT of $x_1[n] = \{1,2,-2\}$ and $x_2[n] = \{1,2,3,-1\}$ respectively.

[2076 Chaitra, 2069 Chaitra]

Solution:

Given,

$x_1[n] = \{1, 2, -2\}$

$x_2[n] = \{1, 2, 3, -1\}$

and, $X_3(k) = X_1(k)X_2(k)$

For 4-point DFT, performing necessary zero padding, we have

$x_1[n] = \{1, 2, -2, 0\}$; N = 4

Now, using convolution property of DFT, we have

$$x_1[n] \textcircled{N} x_2[n] \xleftrightarrow[N]{DFT} X_1(k)X_2(k)$$

Since $X_1(k)$ and $X_2(k)$ are DFT of $x_1[n]$ and $x_2[n]$ respectively and DFT of $x_3[n]$ is given by $X_3(k)$,

$$x_3[n] = x_1[n] \textcircled{N} x_2[n]$$

$$\text{or, } x_3[n] = \sum_{m=0}^{N-1} x_1[m]x_2[n-m]_N$$

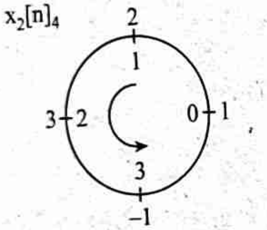
$$= \sum_{m=0}^{3} x_1[m]x_2[n-m]_4$$

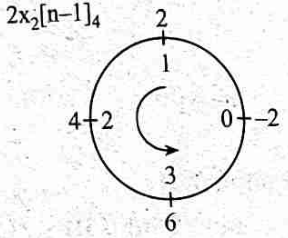
$$= x_1[0]x_2[n]_4 + x_1[1]x_2[n-1]_4 + x_1[2]x_2[n-2]_4 + x_1[3]x_2[n-3]_4$$

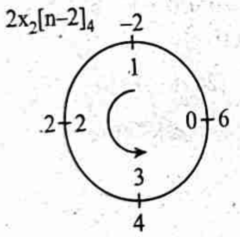
$$= 1 \times x_2[n]_4 + 2x_2[n-1]_4 + (-2)x_2[n-2]_4 + 0$$

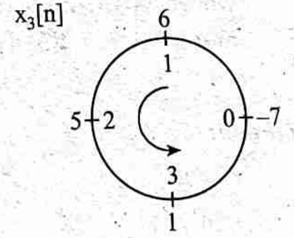
$$\therefore\ x_3[n] = x_2[n]_4 + 2x_2[n-1]_4 - 2x_2[n-2]_4$$

Plotting each elements in circular scale, we get









$$\therefore\ x_3[n] = \{-7, 6, 5, 1\}$$
$$\uparrow$$

Example 7.12:

Find $x_3[n]$ if DFT of $x_3[n]$ is given by $X_3(k) = X_1(k)X_2(k)$ where $X_1(k)$ and $X_2(k)$ are 5-point DFT of $x_1[n] = \{1, -2, 5, 1, 2\}$ and $x_2[n] = \{1, 2, -3, -2\}$ respectively.

[2073 Chaitra]

Solution:

Given,

$x_1[n] = \{1, -2, 5, 1, 2\}$

$x_2[n] = \{1, 2, -3, -2\}$

and, $X_3(k) = X_1(k)X_2(k)$

For 5-point DFT, performing necessary zero padding,

$x_1[n] = \{1, -2, 5, 1, 2\}$

$x_2[n] = \{1, 2, -3, -2, 0\}$

$N = 5$

Now, using convolution property of DTF, we have

$$x_1[n] \textcircled{N} x_2[n] \xleftrightarrow[N]{DFT} X_1(k)X_2(k)$$

Since $X_1(k)$ and $X_2(k)$ are DFT of $x_1[n]$ and $x_2[n]$ respectively and DFT of $x_3[n]$ is given by $X_3(k)$.

$$x_3[n] = x_1[n] \textcircled{N} x_2[n]$$

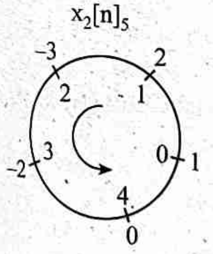
$$\text{or, } x_3[n] = \sum_{m=0}^{N-1} x_1[m]x_2[n-m]_N$$

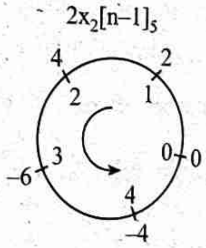
$$= x_1[0]x_2[n]_5 + x_1[1]x_2[n-1]_5 + x_1[2]x_2[n-2]_5 + x_1[3]x_2[n-3]_5 + x_1[4]x_2[n-4]_5$$

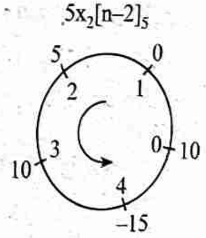
$$= 1x_2[n]_5 + (-2)x_2[n-1]_5 + 5x_2[n-2]_5 + 1x_2[n-3]_5 + 2x_2[n-4]_5$$

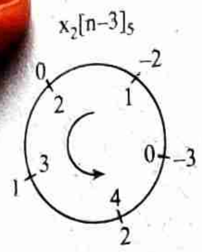
$$\therefore\ x_3[n] = x_2[n]_5 - 2x_2[n-1]_5 + 5x_2[n-2]_5 + x_2[n-3]_5 + 2x_2[n-4]_5$$

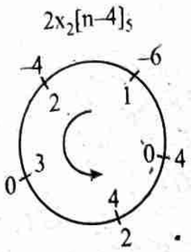
Plotting each element in circular scale, we get

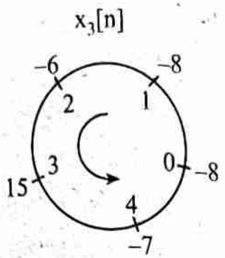












$\therefore\ x_3[n] = \{-8, -8, -6, 15, -7\}$
$\uparrow$

Example 7.13:

Find the linear convolution through circular convolution with paddings of zeros for the following sequences:
$x[n] = \{1, 1, 1, 1\}$ and $h[n] = \{2, 3\}$. [2074 Chaitra]

Solution:

Given,

$x[n] = \{1, 1, 1, 1\}$

$h[n] = \{2, 3\}$

Here,

Length of $x[n] = 4$

Length of $h[n] = 2$

Suppose $y[n]$ is the required linear convolution then the length of $y[n]$ is $4 + 2 - 1 = 5$ (N).

Now, performing necessary zero padding, we have

$x[n] = \{1, 1, 1, 1, 0\}$

$h[n] = \{2, 3, 0, 0, 0\}$

Using circular convolution, we have

$y[n] = x[n] \text{Ⓝ} h[n]$ or $h[n] \text{Ⓝ} x[n]$

To avoid higher number of shifting, we take $h[n] \text{Ⓝ} x[n]$.

$$y[n] = \sum_{m=0}^{N-1} h[m]x[n-m]_N$$

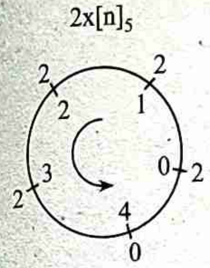
$$= \sum_{m=0}^{4} h[m]x[n-m]_5$$

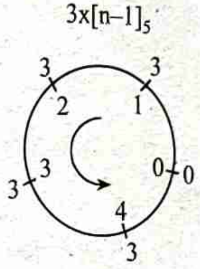
$$= h[0]x[n]_5 + h[1]x[n-1]_5 + h[2]x[n-2]_5 + h[3]x[n-3]_5$$
$$h[4]x[n-4]_5$$

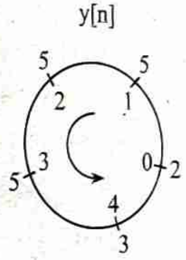
$$= 2 \times x[n]_5 + 3 \times x[n-1]_5 + 0 \times x[n-2]_5 + 0 \times x[n-3]_5$$
$$+ 0 \times x[n-4]_5$$

$\therefore\ y[n] = 2x[n]_5 + 3x[n-1]_5$

Plotting each elements in circular scale, we get







$\therefore\ y[n] = \{2, 5, 5, 5, 3\}$
$\uparrow$

Example 7.14:

Find circular convolution between $x[n] = \{1, 2\}$ and $y[n] = u[n] - u[n-4]$. [2071 Chaitra, 2070 Ashadh]

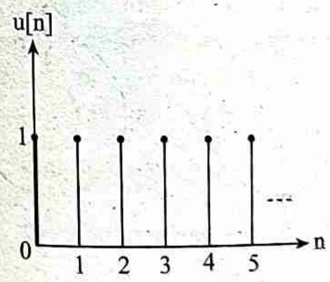
Solution:

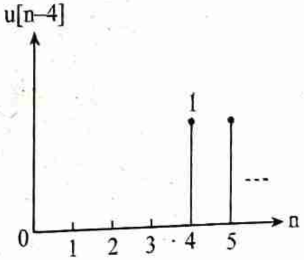
Given,

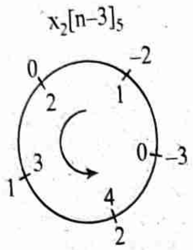
$x[n] = \{1, 2\}$

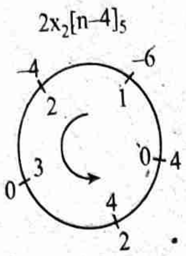
$y[n] = u[n] - u[n-4]$

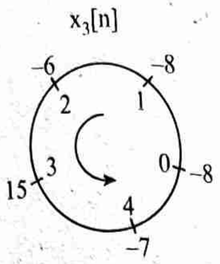
In graphical representation,











$$\therefore\ x_3[n] = \{\underset{\uparrow}{-8}, -8, -6, 15, -7\}$$

Example 7.13:

Find the linear convolution through circular convolution with paddings of zeros for the following sequences:
x[n] = {1, 1, 1, 1} and h[n] = {2, 3}. [2074 Chaitra]

Solution:

Given,

$x[n] = \{1, 1, 1, 1\}$

$h[n] = \{2, 3\}$

Here,

Length of $x[n] = 4$

Length of $h[n] = 2$

Suppose $y[n]$ is the required linear convolution then the length of $y[n]$ is $4 + 2 - 1 = 5$ (N).

Now, performing necessary zero padding, we have

$x[n] = \{1, 1, 1, 1, 0\}$

$h[n] = \{2, 3, 0, 0, 0\}$

Using circular convolution, we have

$y[n] = x[n] \text{Ⓝ} h[n]$ or $h[n] \text{Ⓝ} x[n]$

To avoid higher number of shifting, we take $h[n] \text{Ⓝ} x[n]$.

$$y[n] = \sum_{m=0}^{N-1} h[m]x[n-m]_N$$

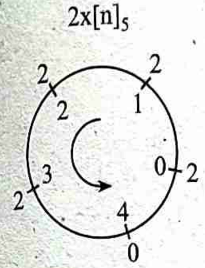
$$= \sum_{m=0}^{4} h[m]x[n-m]_5$$

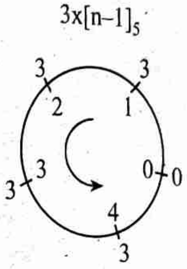
$$= h[0]x[n]_5 + h[1]x[n-1]_5 + h[2]x[n-2]_5 + h[3]x[n-3]_5\ h[4]x[n-4]_5$$

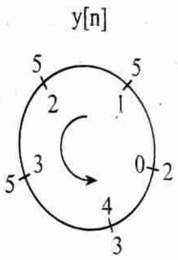
$$= 2 \times x[n]_5 + 3 \times x[n-1]_5 + 0 \times x[n-2]_5 + 0 \times x[n-3]_5 + 0 \times x[n-4]_5$$

$$\therefore\ y[n] = 2x[n]_5 + 3x[n-1]_5$$

Plotting each elements in circular scale, we get







$$\therefore\ y[n] = \{\underset{\uparrow}{2}, 5, 5, 5, 3\}$$

Example 7.14:

Find circular convolution between x[n] = {1, 2} and y[n] = u[n] – u[n – 4]. [2071 Chaitra, 2070 Ashadh]

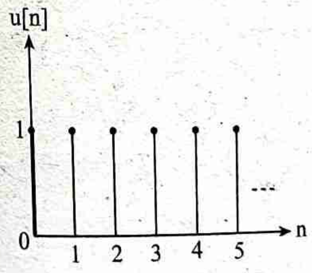
Solution:

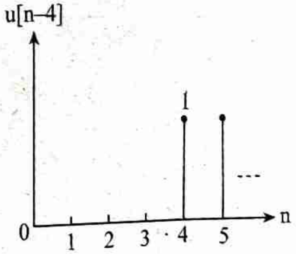
Given,

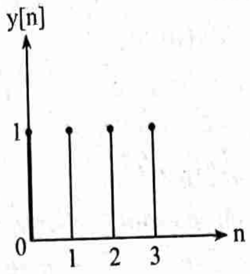
$x[n] = \{1, 2\}$

$y[n] = u[n] - u[n-4]$

In graphical representation,







$\therefore$ $y[n] = \{1, 1, 1, 1\}$

Here, $N = 4$.

Performing necessary zero paddings, we have

$x[n] = \{1, 2, 0, 0\}$

$y[n] = \{1, 1, 1, 1\}$

Using circular convolution, we get

$$z[n] = x[n] \text{ Ⓝ } y[n]$$

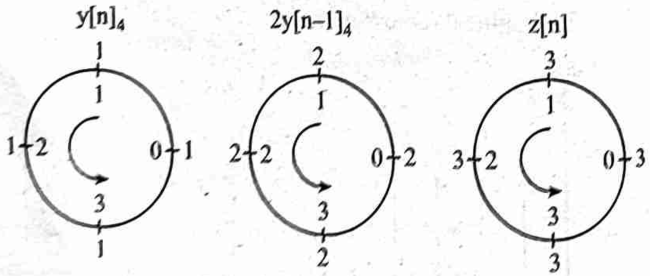
$$= \sum_{m=0}^{N-1} x[m]y[n-m]_N$$

$$= \sum_{m=0}^{3} x[m]y[n-m]_4$$

$$= x[0]y[n]_4 + x[1]y[n-1]_4 + x[2]y[n-2]_4 + x[3]y[n-3]_4$$

$$\therefore \; z[n] = 1 \times y[n]_4 + 2 \times y[n-1]_4 + 0 \times y[n-2]_4 + 0 \times y[n-3]_4$$

Putting each elements in circular scale, we get



$$\therefore \; z[n] = \{\underset{\uparrow}{3}, 3, 3, 3\}$$

7.2 Fast Fourier Transform (FFT) Algorithm (Decimation in Time Algorithm, Decimation in Frequency Algorithm)

The Discrete Fourier Transform (DFT) plays an important role in many applications of digital signal processing including linear filtering, correlation analysis, and spectrum analysis. So it is important to compute DFT using efficient algorithm. *Fast Fourier Transform (FFT)* is an algorithm that computes the DFT of a sequence quickly and efficiently, reducing the computational complexity to $\frac{N}{2} \log_2 N$.

In direct computation of the DFT, the DFT for a complex-valued sequence x(n) of N points is given by

$$X(k) = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

$$\text{or, } X(k) = \sum_{n=0}^{N-1} x[n]W_N^{kn}$$

Here, we can see that direct computation of the DFT requires N complex multiplications for each k and N^2 complex multiplications for all k. Similarly, it requires N – 1 complex additions for each k and N(N – 1) complex additions for all k. Therefore, the computational requirement to compute DFT directly is very high.

Direct computation of the DFT is basically inefficient primarily because it does not exploit the symmetry and periodicity properties of the phase factor W_N. In particular, these two properties are:

Symmetry property: $W_N^{k+N/2} = -W_N^k$

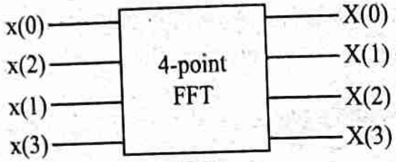
Periodicity property: $W_N^{k+N} = W_N^k$

Fast Fourier Transform (FFT) algorithms exploit these two basic properties of the phase factor.

Radix-2 FFT

It is by for the most widely used FFT algorithm. The radix-2 representation to calculate N-point FFT is $N = 2^v$: where v represents the number of stages in FFT. Suppose N = 8, then we have $N = 2^3$. Hence, there will be 3 stages and it is called 8-point FFT. Radix-2 FFT algorithm can be implemented by two ways:

I. Decimation in time (DIT) FFT



DIT

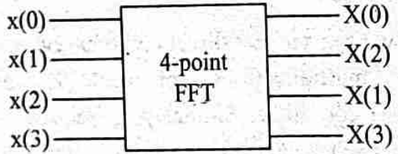
Before decimation;

x(0) x(1) x(2) x(3)

After decimation;

x(0) x(2) x(1) x(3)

II. Decimation in frequency (DIF) FFT



DIF

Before decimation:

X(0) X(1) X(2) X(3)

After decimation:

X(0) X(2) X(1) X(3)

8-Point FFT

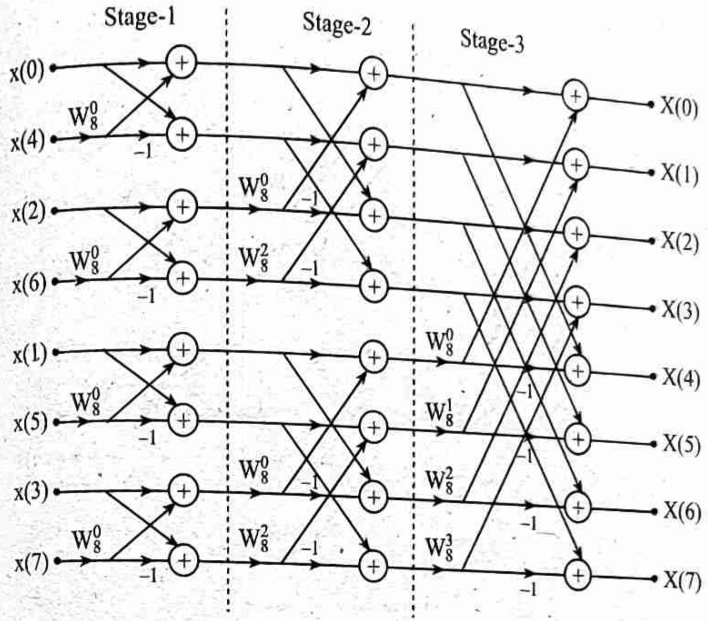
Here, N = 8 so, $N = 2^3$ means v = 3

So the number of stages are 3.

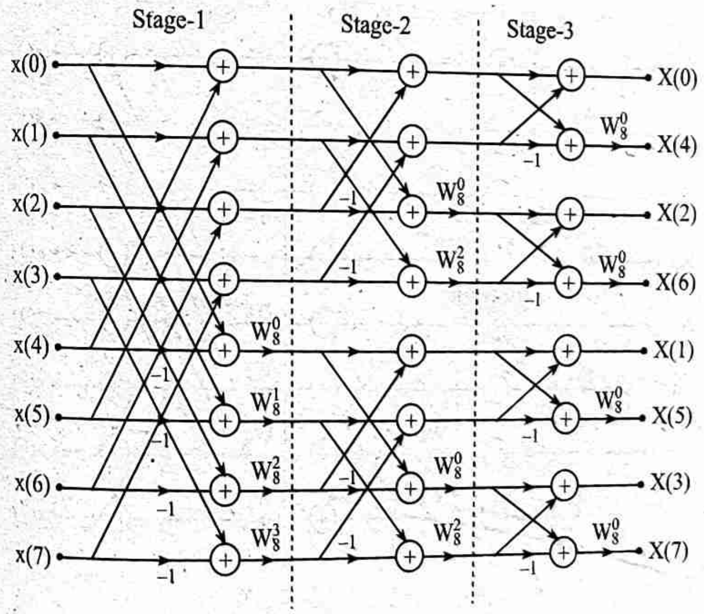
Decimation process:

x(0) x(1) x(2) x(3) x(4) x(5) x(6) x(7)
x(0) x(2) x(4) x(6) x(1) x(3) x(5) x(7)
x(0) x(4) x(2) x(6) x(1) x(5) x(3) x(7)

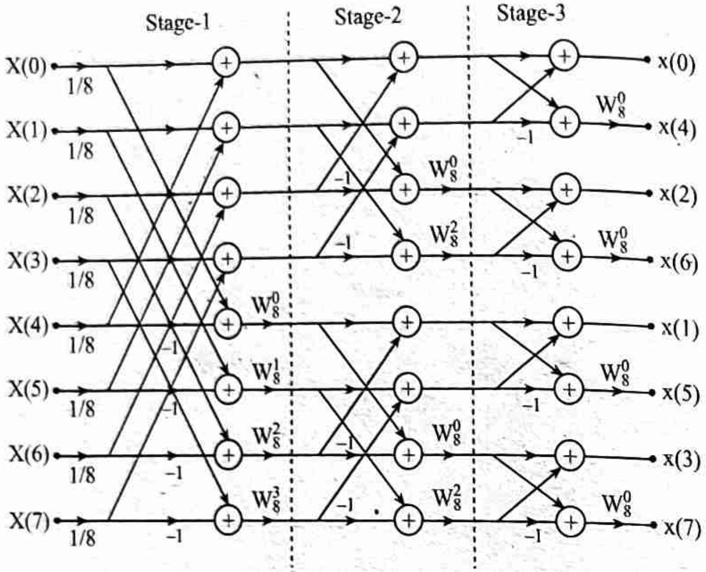
8-Point Radix-2 DIT FFT



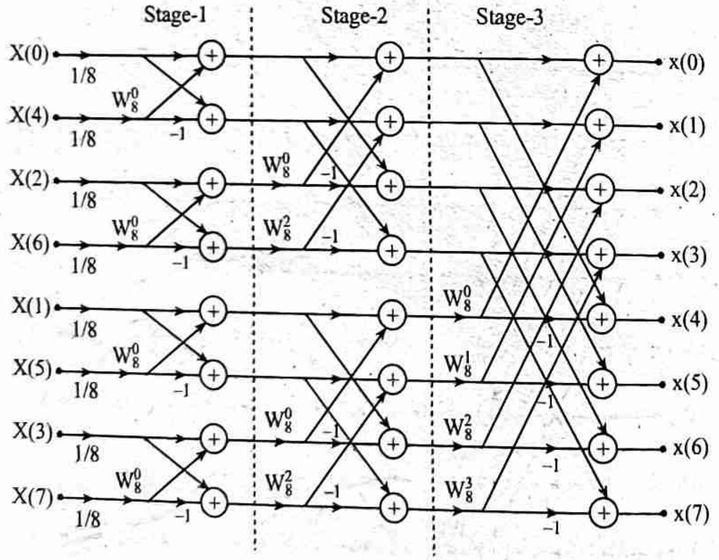
8-Point Radix-2 DIF FFT



8-Point Radix-2 DIT IFFT



8-point Radix-2 DIF IFFT



Example 7.15:

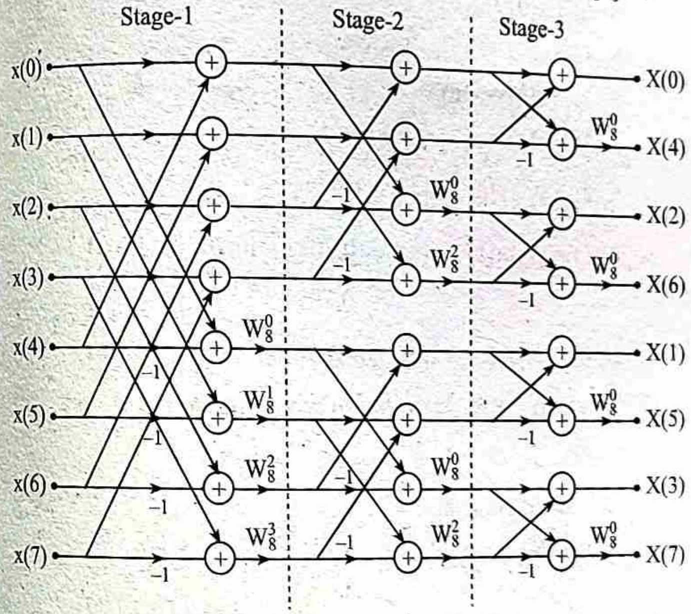
Compute 8-point DIF-FFT of sequence x[n] = {2, 1, 2, 1, 1, 2, 1, 2}. **[2080 Bhadra]**

Solution:

Given, x[n] = {2, 1, 2, 1, 1, 2, 1, 2}

where,

x[0] = 2	x[1] = 1	x[2] = 2	x[3] = 1
x[4] = 1	x[5] = 2	x[6] = 1	x[7] = 2



Now, we know

$$W_N^{kn} = e^{-j2\pi kn/N}$$

So,

$$W_8^0 = e^{-j2\pi \times 0/8} = e^0 = 1$$

$$W_8^1 = e^{-j2\pi \times 1/8} = \cos\left(\frac{\pi}{4}\right) - j\sin\left(\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} = 0.7071 - j0.7071$$

$$W_8^2 = e^{-j2\pi\times2/8} = \cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right) = -j$$

$$W_8^3 = e^{-j2\pi\times3/8} = \cos\left(\frac{3\pi}{4}\right) - j\sin\left(\frac{3\pi}{4}\right)$$

$$= -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$= -0.7071 - j0.7071$$

Output of stage 1:

$S_1(0) = x[0] + x[4] = 2 + 1 = 3$

$S_1(1) = x[1] + x[5] = 1 + 2 = 3$

$S_1(2) = x[2] + x[6] = 2 + 1 = 3$

$S_1(3) = x[3] + x[7] = 1 + 2 = 3$

$S_1(4) = W_8^0[x[0] - x[4]] = 1(2 - 1) = 1$

$S_1(5) = W_8^1[x[1] - x[5]] = (0.7071 - j0.7071)\,(1 - 2)$

$= -0.7071 + j0.7071$

$S_1(6) = W_8^2[x[2] - x[6]] = (-j)(2 - 1) = -j$

$S_1(7) = W_8^3[x[3] - x[7]] = (-0.7071 - j0.7071)\,(1 - 2)$

$= 0.7071 + j0.7071$

Output of stage 2:

$S_2(0) = S_1(0) + S_1(2) = 3 + 3 = 6$

$S_2(1) = S_1(1) + S_1(3) = 3 + 3 = 6$

$S_2(2) = W_8^0[S_1(0) - S_1(2)] = 1(3 - 3) = 0$

$S_2(3) = W_8^2[S_1(1) - S_1(3)] = -j(3 - 3) = 0$

$S_2(4) = S_1(4) + S_1(6) = 1 + (-j) = 1 - j$

$S_2(5) = S_1(5) + S_1(7)$

$= (-0.7071 + j0.7071) + (0.7071 + j0.7071)$

$= j1.4142$

$S_2(6) = W_8^0[S_1(4) - S_1(6)] = 1[1 - (-j)] = 1 + j$

$S_2(7) = W_8^2[S_1(5) - S_1(7)]$

$= (-j)[(-0.7071 + j0.7071) - (0.7071 + j0.7071)]$

$= -j \times -1.414$

$= j1.4142$

Final output:

$X(0) = S_2(0) + S_2(1) = 6 + 6 = 12$

$X(4) = W_8^0[S_2(0) - S_2(1)] = 1(6 - 6) = 0$

$X(2) = S_2(2) + S_2(3) = 0 + 0 = 0$

$X(6) = W_8^0[S_2(2) - S_2(3)] = 1(0 - 0) = 0$

$X(1) = S_2(4) + S_2(5) = 1 - j + j1.4142 = 1 + j0.4142$

$X(5) = W_8^0[S_2(4) - S_2(5)] = 1(1 - j - j1.4142) = 1 - j2.4142$

$X(3) = S_2(6) + S_2(7) = 1 + j + j1.4142 = 1 + j2.4142$

$X(7) = W_8^0[S_2(6) - S_2(7)] = 1(1 + j - j1.4142) = 1 - j0.4142$

Thus, we have

$X(k) = \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\}$

$\therefore\ X(k) = \{12, 1 + j0.4142, 0, 1 + j2.4142, 0, 1 - j2.4142, 0, 1 - j0.4142\}$

Example 7.16:

Find 8-point DFT of sequence x[n] = {1, 2, 4, 3, 5, –1, 3} using Decimation in Frequency Fast Fourier Transform (DIFFFT) algorithm. **[2079 Baishakh]**

Solution:

Given, $x[n] = \{1, 2, 4, 3, 5, -1, 3\}$

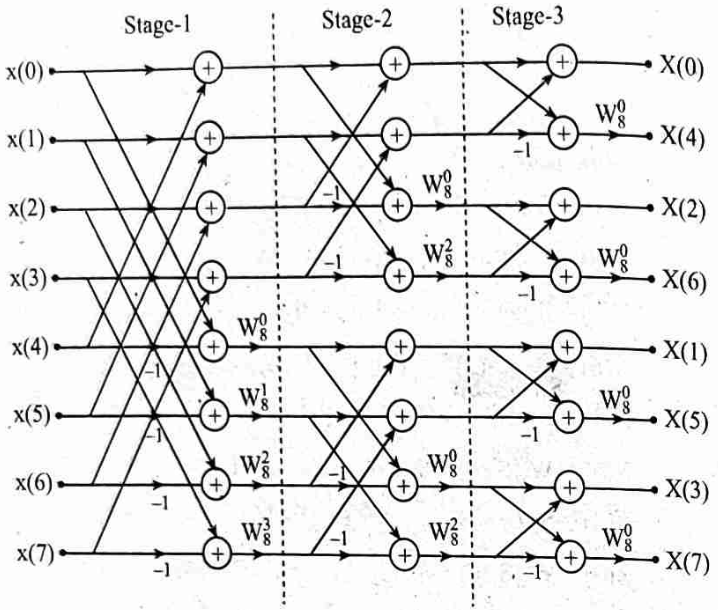
Performing necessary zero paddings, we have

$x[n] = \{1, 2, 4, 3, 5, -1, 3, 0\}$

where,

$x[0] = 1,$	$x[1] = 2,$	$x[2] = 4,$	$x[3] = 3,$
$x[4] = 5,$	$x[5] = -1,$	$x[6] = 3,$	$x[7] = 0$

Here,



Now, we know

$W_N^{kn} = e^{-j2\pi kn/N}$

So,

$W_8^0 = e^{-j2\pi.0/8} = 1$

$W_8^1 = e^{-j2\pi.1/8} = 0.7071 - j0.7071$

$W_8^2 = e^{-j2\pi.2/8} = -j$

$W_8^3 = e^{-j2\pi.3/8} = -0.7071 - j0.7071$

Output of stage 1:

$S_1(0) = x[0] + x[4] = 1 + 5 = 6$

$S_1(1) = x[1] + x[5] = 2 + (-1) = 1$

$S_1(2) = x[2] + x[6] = 4 + 3 = 7$

$S_1(3) = x[3] + x[7] = 3 + 0 = 3$

$S_1(4) = W_8^0[x[0] - x[4]] = 1(1 - 5) = -4$

$S_1(5) = W_8^1[x[1] - x[5]] = (0.7071 - j0.7071) - (2 - (-1))$
$= 3(0.7071 - j0.7071)$
$= 2.1213 - j2.1213$

$S_1(6) = W_8^2[x[2] - x[6]] = (-j)(4 - 3) = -j$

$S_1(7) = W_8^3[x[3] - x[7]] = (-0.7071 - j0.7071)(3 - 0)$
$= 3(-0.7071 - j0.7071)$
$= -2.1213 - j2.1213$

Output of stage 2:

$S_2(0) = S_1(0) + S_1(2) = 6 + 7 = 13$

$S_2(1) = S_1(1) + S_1(3) = 1 + 3 = 4$

$S_2(2) = W_8^0[S_1(0) - S_1(2)] = 1(6 - 7) = -1$

$S_2(3) = W_8^2[S_1(1) - S_1(3)] = -j(1 - 3) = -j \times -2 = 2j$

$S_2(4) = S_1(4) + S_1(6) = -4 + (-j) = -4 - j$

$S_2(5) = S_1(5) + S_1(7)$
$= (2.1213 - j2.1213) + (-2.1213 - j2.1213)$
$= -j4.2426$

$S_2(6) = W_8^0[S_1(4) - S_1(6)] = 1[-4 - (-j)] = -4 + j$

$S_2(7) = W_8^2[S_1(5) - S_1(7)]$
$= (-j)[(2.1213 - j2.1213) - (-2.1213 - j2.1213)]$
$= -j4.2426$

Final output:

$X(0) = S_2(0) + S_2(1) = 13 + 4 = 17$

$X(4) = W_8^0[S_2(0) - S_2(1)] = 1(13 - 4) = 9$

$X(2) = S_2(2) + S_2(3) = -1 + 2j$

$X(6) = W_8^0[S_2(2) - S_2(3)] = 1(-1 - 2j) = -1 - 2j$

$X(1) = S_2(4) + S_2(5) = (-4 - j) + (-j4.2426)$
$= -4 - j5.2426$

$X(5) = W_8^0[S_2(4) - S_2(5)] = 1[(-4 - j) - (-j4.2426)]$
$= -4 + j3.2426$

$$X(3) = S_2(6) + S_2(7) = (-4 + j) + (-j4.2426)$$
$$= -4 - j3.2426$$

$$X(7) = W_8^0[S_2(6) - S_2(7)] = 1[(-4 + j) - (-j4.2426)]$$
$$= -4 + j5.2426$$

Thus, we have,

$X(k) = \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\}$

$\therefore$ $X(k) = \{17, -4 - j5.2426, -1 + 2j, -4 - j3.2426, 9, -4 + j3.2426, -1 - 2j, -4 + j5.2426\}$

Example 7.17:

Find 8-point DFT of sequence x[n] = {1, 1, 0, 0, 1, 1, 2} using Decimation in Time Fast Fourier Transform (DITFFT) algorithm. [2080 Baishakh]

Solution:

Given, $x[n] = \{1, 1, 0, 0, 1, 1, 2\}$

Performing necessary zero paddings, we have

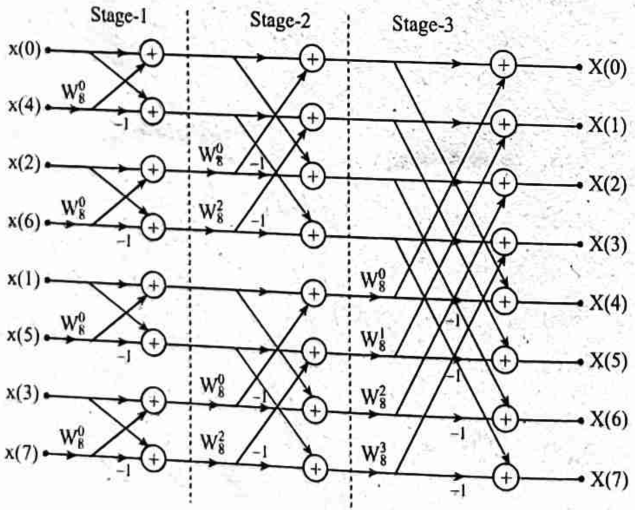
$x[n] = \{1, 1, 0, 0, 1, 1, 2, 0\}$

where,

$x[0] = 1,\quad x[1] = 1,\quad x[2] = 0,\quad x[3] = 0,$

$x[4] = 1,\quad x[5] = 1,\quad x[6] = 2,\quad x[7] = 0$

Here,



Now, we know

$$W_N^{kn} = e^{-j2\pi kn/N}$$

So,

$$W_8^0 = e^{-j2\pi\times 0/8} = 1$$

$$W_8^1 = e^{-j2\pi\times 1/8} = 0.7071 - j0.7071$$

$$W_8^2 = e^{-j2\pi\times 2/8} = -j$$

$$W_8^3 = e^{-j2\pi\times 3/8} = -0.7071 - j0.7071$$

Output of stage 1:

$$S_1(0) = x[0] + W_8^0x[4] = 1 + 1 \times 1 = 2$$

$$S_1(1) = x[0] - W_8^0x[4] = 1 - 1 \times 1 = 0$$

$$S_1(2) = x[2] + W_8^0x[6] = 0 + 1 \times 2 = 2$$

$$S_1(3) = x[2] - W_8^0x[6] = 0 - 1 \times 2 = -2$$

$$S_1(4) = x[1] + W_8^0x[5] = 1 + 1 \times 1 = 2$$

$$S_1(5) = x[1] - W_8^0x[5] = 1 - 1 \times 1 = 0$$

$$S_1(6) = x[3] + W_8^0x[7] = 0 + 1 \times 0 = 0$$

$$S_1(7) = x[3] - W_8^0x[7] = 0 - 1 \times 0 = 0$$

Output of stage 2:

$$S_2(0) = S_1(0) + W_8^0S_1(2) = 2 + 1 \times 2 = 4$$

$$S_2(1) = S_1(1) + W_8^2S_1(3) = 0 + (-j) \times (-2) = 2j$$

$$S_2(2) = S_1(0) - W_8^0S_1(2) = 2 - 1 \times 2 = 0$$

$$S_2(3) = S_1(1) - W_8^2S_1(3) = 0 - (-j) \times (-2) = -2j$$

$$S_2(4) = S_1(4) + W_8^0S_1(6) = 2 + 1 \times 0 = 2$$

$$S_2(5) = S_1(5) + W_8^2S_1(7) = 0 + (-j) \times 0 = 0$$

$S_2(6) = S_1(4) - W_8^0 S_1(6) = 2 - 1 \times 0 = 2$

$S_2(7) = S_1(5) - W_8^2 S_1(7) = 0 - (-j) \times 0 = 0$

Final output:

$X(0) = S_2(0) + W_8^0 S_2(4) = 4 + 1 \times 2 = 6$

$X(1) = S_2(1) + W_8^1 S_2(5) = 2j + (0.7071 - j0.7071) \times 0 = 2j$

$X(2) = S_2(2) + W_8^2 S_2(6) = 0 + (-j)\times 2 = -2j$

$X(3) = S_2(3) + W_8^3 S_2(7) = -2j + (-0.7071 - j0.7071)\times 0 = -2j$

$X(4) = S_2(0) - W_8^0 S_2(4) = 4 - 1\times 2 = 2$

$X(5) = S_2(1) - W_8^1 S_2(5) = 2j - (0.7071 - j0.7071) \times 0 = 2j$

$X(6) = S_2(2) - W_8^2 S_2(6) = 0 - (-j) \times 2 = 2j$

$X(7) = S_2(3) - W_8^3 S_2(7) = -2j - (-0.7071 - j0.7071) \times 0 = -2j$

Thus, we have

$X(k) = \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\}$

$\therefore\ X(k) = \{6, 2j, -2j, -2j, 2, 2j, 2j, -2j\}$

Example 7.18:

Find the 8-point DFT of the following sequence using radix-2 DITFFT algorithm.

$x[n] = \{1, 2j, 3, 4j, 0, 0, -j, 0\}$. **[2079 Bhadra]**

Solution:

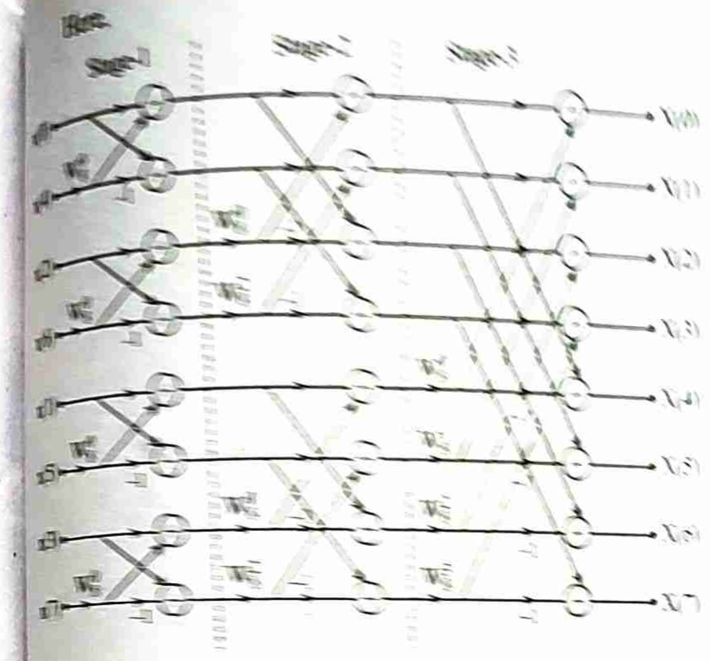
Given,

$x[n] = \{1, 2j, 3, 4j, 0, 0, -j, 0\}$

where,

$x[0] = 1,$	$x[1] = 2j,$	$x[2] = 3,$	$x[3] = 4j,$
$x[4] = 0,$	$x[5] = 0,$	$x[6] = -j,$	$x[7] = 0$

Here,



Now, we know

$W_N^{kn} = e^{-j2\pi kn/N}$

So,

$W_8^0 = e^{\text{[illegible]}} = 1$

$W_8^1 = e^{\text{[illegible]}} = 0.7071 - j0.7071$

$W_8^2 = e^{\text{[illegible]}} = -j$

$W_8^3 = e^{\text{[illegible]}} = -0.7071 - j0.7071$

Output of stage 1:

$S_1(0) = x[0] + W_8^0 x[4] = 1 + 1 \times 0 = 1$

$S_1(1) = x[0] - W_8^0 x[4] = 1 - 1 \times 0 = 1$

$S_1(2) = x[2] + W_8^0 x[6] = 3 + 1 \times (-j) = 3 - j$

$S_1(3) = x[2] - W_8^0 x[6] = 3 - 1 \times (-j) = 3 + j$

$S_1(4) = x[1] + W_8^0 x[5] = 2j + 1 \times 0 = 2j$

$S_1(5) = x[1] - W_8^0 x[5] = 2j - 1 \times 0 = 2j$

$S_1(6) = x[3] + W_8^0 x[7] = 4j + 1 \times 0 = 4j$

$S_1[7] = x[3] - W_8^0 x[7] = 4j - 1 \times 0 = 4j$

Output of stage 2:

$S_2(0) = S_1(0) + W_8^0 S_1(2) = 1 + 1 \times \ (3 - j) = 4 - j$

$S_2(1) = S_1(1) + W_8^2 S_1(3) = 1 + (-j)(3 + j) = 1 - 3j - j^2 = 2 - 3j$

$S_2(2) = S_1(0) - W_8^0 S_1(2) = 1 - 1 \times \ (3 - j) = -2 + j$

$S_2(3) = S_1(1) - W_8^2 S_1(3) = 1 - (-j)(3 + j) = 1 + 3j + j^2 = 3j$

$S_2(4) = S_1(4) + W_8^0 S_1(6) = 2j + 1 \times 4j = 6j$

$S_2(5) = S_1(5) + W_8^2 S_1(7) = 2j + (-j)4j = 4 + 2j$

$S_2(6) = S_1(4) - W_8^0 S_1(6) = 2j - 1 \times 4j = -2j$

$S_2(7) = S_1(5) - W_8^2 S_1(7) = 2j - (-j)4j = -4 + 2j$

Final output:

$X(0) = S_2(0) + W_8^0 S_2(4) = (4 - j) + 1 \times 6j = 4 + 5j$

$X(1) = S_2(1) + W_8^1 S_2(5)$

$= (2 - 3j) + (0.7071 - j0.7071) \times \ (4 + 2j)$

$= (2 - 3j) + (4.242 - j1.414)$

$= 6.242 - j4.414$

$X(2) = S_2(2) + W_8^2 S_2(6) = (-2 + j) + (-j)(-2j) = -4 + j$

$X(3) = S_2(3) + W_8^3 S_2(7) = 3j + (-0.7071 - j0.7071)(-4 + 2j)$

$= 4.242 + j4.414$

$X(4) = S_2(0) - W_8^0 S_2(4) = (4 - j) - 1 \times 6j = 4 - 7j$

$X(5) = S_2(1) - W_8^1 S_2(5)$

$= (2 - 3j) - (0.7071 - j0.7071) \times \ (4 + 2j)$

$= -2.242 - j1.586$

$X(6) = S_2(2) - W_8^2 S_2(6) = (-2 + j) - (-j)(-2j) = j$

$X(7) = S_2(3) - W_8^3 S_2(7) = 3j - (-0.7071 - j0.7071)\ (-4 + 2j)$

$= -4.242 + j1.586$

Thus, we have

$X(k) = \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\}$

$\therefore \ X(k) = \{4+5j, 6.242-j4.414, -4+j, 4.242+j4.414, 4-7j, -2.242-j1.586, j, -4.242 + j1.586\}$

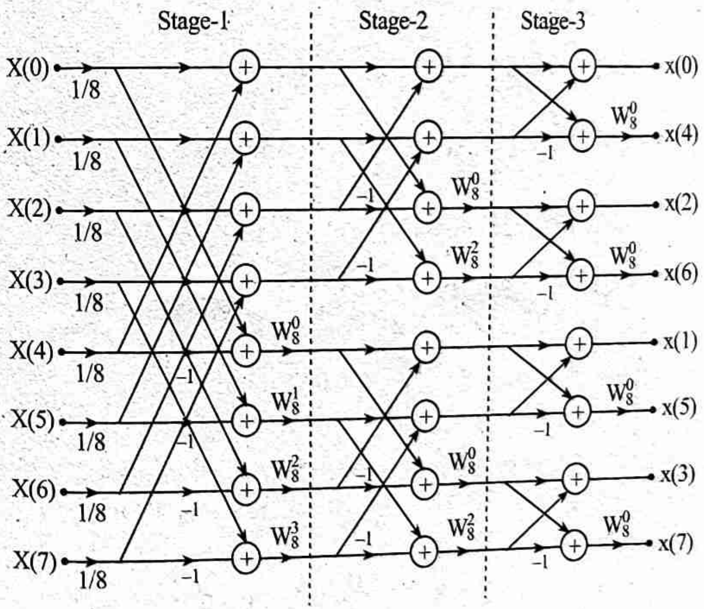
Example 7.19:

Find the 8-point DIT IDFT for X(k) = {2, 0.5–j1.207, 0, 0.5–j0.207, 0, 0.5+j0.207, 0, 0.5 + j1.21}.

Solution:

$X(0) = 2, X(1) = 0.5 - j1.207, X(2) = 0, X(3) = 0.5 - j0.207,$

$X(4) = 0, X(5) = 0.5 + j0.207, X(6) = 0, X(7) = 0.5 + j1.21$



We know,

$W_N^{kn} = e^{-j2\pi kn/N}$

So, $W_8^0 = e^{-j2\pi\times 0/8} = 1$

$W_8^{-1} = e^{+j2\pi\times 1/8} = +0.707 + j0.707$

$W_8^{-2} = e^{+j2\pi\times 2/8} = +j$

$W_8^{-3} = e^{+j2\pi\times 3/8} = -0.707 + j\,0.707$

Output of stage 1:

$S_1(0) = \frac{1}{8}[X(0) + X(4)] = \frac{1}{4} = 0.25$

$S_1(1) = \frac{1}{8}[X(1) + X(5)] = \frac{1}{8}(1 - j) = 0.125 - j0.125$

$S_1(2) = \frac{1}{8}[X(2) + X(6)] = \frac{1}{8}\times 0 = 0$

$S_1(3) = \frac{1}{8}[X(3) + X(7)] = \frac{1}{8}(1 + j) = 0.125 + j0.125$

$S_1(4) = \frac{1}{8}[X(0) - X(4)] = \frac{1}{8}\times 2 = 0.25$

$S_1(5) = \frac{1}{8}[X(1) - X(5)] = -\frac{1}{8}\times j1.414 = -j0.1768$

$S_1(6) = \frac{1}{8}[X(2) - X(6)] = \frac{1}{8}\times 0 = 0$

$S_1(7) = \frac{1}{8}[X(3) - X(7)] = -\frac{1}{8}\times j1.417 = 0.1771$

Output of stage 2:

$S_2(0) = S_1(0) + S_1(2) = 0.25 + 0 = 0.25$

$S_2(1) = S_1(1) + S_1(3) = 0.125 - j0.125 + 0.125 + j0.125 = 0.25$

$S_2(2) = S_1(0) - S_1(2) = 0.25 - 0 = 0.25$

$S_2(3) = S_1(1) - S_1(3) = -j0.25$

$S_2(4) = S_1(4)W_8^0 + S_1(6)W_8^{-2} = 0.25 + 0 = 0.25$

$S_2(5) = S_1(5)W_8^{-1} + S_1(7)W_8^{-3}$

$= -j0.1768(0.707 + j0.707) - j0.1771(-0.707 + j0.707)$

$= 0.25$

$S_2(6) = S_1(4)W_8^0 - S_1(6)W_8^{-2} = 0.25\times 1 - 0\times j = 0.25$

$S_2(7) = S_1(5)W_8^{-1} - S_1(7)W_8^{-3}$

$= -j0.1768(0.707 + j0.707) - (-j0.1771)(-0.707 + j0.707)$

$= -j0.25$

Final output:

$x[0] = S_2(0) + S_2(1) = 0.25 + 0.25 = 0.5$

$x[4] = S_2(0) - S_2(1) = 0.25 - 0.25 = 0$

$x[2] = S_2(2)W_8^0 + S_2(3)W_8^{-2}$

$= (0.25\times 1) + (-j0.25)\times(+j)$

$= 0.5$

$x[6] = S_2(2)W_8^0 - S_2(3)W_8^{-2}$

$= (0.25\times 1) - (-j0.25)\times(+j) = 0$

$x[1] = S_2(4) + S_2(5) = 0.25 + 0.25 = 0.5$

$x[5] = S_2(4) - S_2(5) = 0.25 - 0.25 = 0$

$x[3] = S_2(6)W_8^0 + S_2(7)W_8^{-2}$

$= (0.25\times 1) + (-j0.25)\times(+j) = 0.5$

$x[7] = S_2(6)W_8^0 - S_2(7)W_8^{-2}$

$= (0.25\times 1) - (-j0.25)\times(+j) = 0$

$\therefore\ x[n] = \{0.5, 0.5, 0.5, 0.5, 0, 0, 0, 0\}$

Example 7.20:

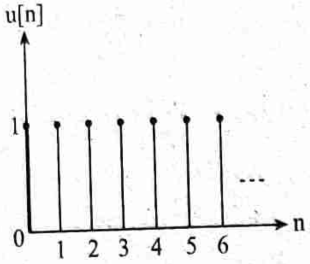
Find the 8-point DFT of $x[n] = u[n] - u[n-4]$ using FFT DIT algorithm. **[2078 Bhadra]**

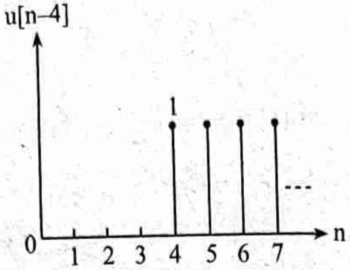
Solution:

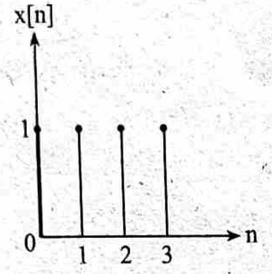
Given,

$x[n] = u[n] - u[n-4]$

In graphical representation,







$\therefore$ $x[n] = \{1, 1, 1, 1\}$

Performing necessary zero paddings, we have

$x[n] = \{1, 1, 1, 1, 0, 0, 0, 0\}$

where,

$x[0] = 1$	$x[1] = 1$	$x[2] = 1$	$x[3] = 1$
$x[4] = 0$	$x[5] = 0$	$x[6] = 0$	$x[7] = 0$

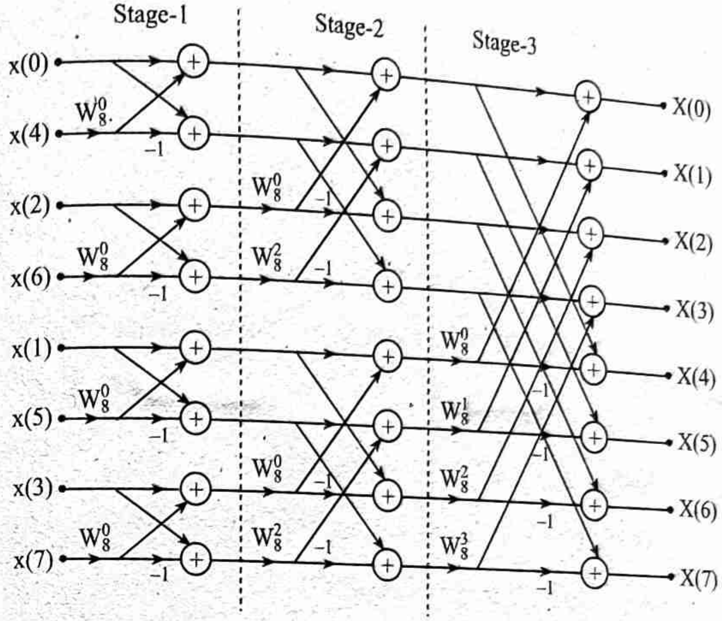
Now this question can be solved similarly as previous question using DIT FFT algorithm.

Example 7.21:

Draw the butterfly diagram of 8-point DFT of a sequence of $x[n] = n + 1$ using Decimation in Time FFT algorithm.

[2075 Ashwin]

Solution:



Given, $x[n] = n + 1$

So,

$x[0] = 0 + 1 = 1$

$x[1] = 1 + 1 = 2$

$x[2] = 2 + 1 = 3$

$x[3] = 3 + 1 = 4$

$x[4] = 4 + 1 = 5$

$x[5] = 5 + 1 = 6$

$x[6] = 6 + 1 = 7$

$x[7] = 7 + 1 = 8$

$\therefore$ $x[n] = \{1, 2, 3, 4, 5, 6, 7, 8\}$

Now this question can be solved similarly as previous questions using DIT FFT algorithm.

Example 7.22:

If $X_1(k)$ and $X_2(k)$ are the 5-point DFT of $x_1(n) = 3^n$, $0 \le n \le 3$ and $x_2(n) = 2^n$, $0 \le n \le 4$. Find $x_3(n)$ if $X_3(k) = X_1(k)X_2(k)$. **[2081 Bhadra]**

Solution:

$x_1[n] = 3^n, 0 \le n \le 3$

$\Rightarrow x_1[n] = \{1, 3, 9, 27\}$

and, $x_2[n] = 2^n, 0 \le n \le 4$

$\Rightarrow x_2[n] = \{1, 2, 4, 8, 16\}$

Also,

$X_3(k) = X_1(k)X_2(k)$ and $N = 5$

For 5-point DFT, performing necessary zero padding, we have,

$x_1[n] = \{1, 3, 9, 27, 0\}$

$x_2[n] = \{1, 2, 4, 8, 16\}$

Now, using convolution property of DFT, we have,

$$x_1[n] \; \textcircled{N} \; x_2[n] \xleftrightarrow[N]{\text{DFT}} X_1(k)X_2(k)$$

Since, $X_1(k)$ and $X_2(k)$ are DFT of $x_1[n]$ and $x_2[n]$ respectively and DFT of $x_3[n]$ is given by $X_3(k)$,

$$x_3[n] = x_1[n] \; \textcircled{N} \; x_2[n]$$

$$\text{or, } x_3[n] = \sum_{m=0}^{N-1} x_1[m]x_2[n-m]_N$$

$$= \sum_{m=0}^{4} x_1[m]x_2[n-m]_5$$

$$= x_1[0]x_2[n]_5 + x_1[1]x_2[n-1]_5 + x_1[2]x_2[n-2]_5$$

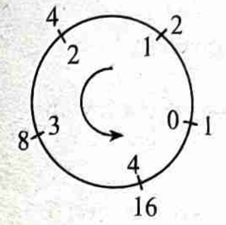
$$+ x_1[3]x_2[n-3]_5 + x_1[4]x_2[n-4]_5$$

$$= 1 \times x_2[n]_5 + 3x_2[n-1]_5 + 9x_2[n-2]_5 + 27x_2[n-3]_5 + 0$$

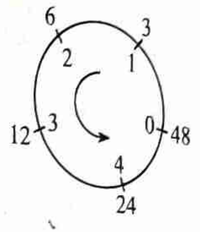
$$\therefore \; x_3[n] = x_2[n]_5 + 3x_2[n-1]_5 + 9x_2[n-2]_5 + 27x_2[n-3]_5$$

Plotting each elements in circular scale, we get

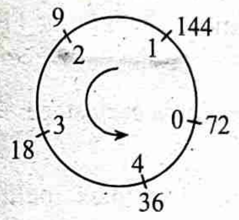
$x_2[n]_5$



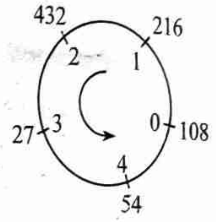
$3x_2[n-1]_5$



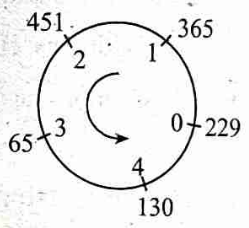
$9x_2[n-2]_5$



$27x_2[n-3]_5$



$x_3[n]$



$$\therefore \; x_3[n] = \{229, 365, 451, 65, 130\}$$
$$\uparrow$$

7.3 Computational Complexity of FFT Algorithm

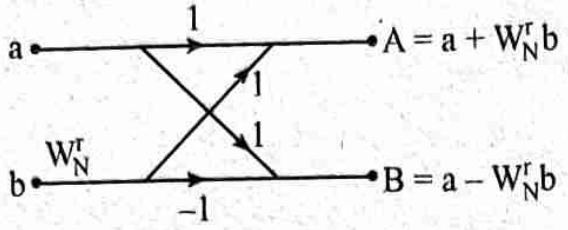
The DFT of a complex-valued sequence x(n) of N points is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$$

$$\text{or, } X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn} \text{; where } W_N = e^{-j2\pi/N}$$

Here, we can see that the direct computation of the DFT requires N^2 complex multiplications and $N(N - 1)$ complex additions for completion.

The butterfly diagram of FFT is given by .



Here, we can see that each butterfly has one complex multiplication and two complex additions. In 8-point FFT, there are three stages and four butterfly in each stage. Hence, N-point FFT will have $\log_2 N$ stages and $\frac{N}{2}$ butterfly in each stage.

Therefore, $\frac{N}{2}$ butterfly will have $\frac{N}{2}$ complex multiplication and N complex addition. So, $\log_2 N$ stages will have $\frac{N}{2}\log_2 N$ complex multiplications and $N\log_2 N$ complex additions. Hence, N-point FFT will have $\frac{N}{2}\log_2 N$ complex multiplications and $N\log_2 N$ complex additions.

Comparison of Direct Computation of DFT and using FFT

Number of points (N)	Direct computation		Using FFT	
	Complex multiplications (N^2)	Complex additions [$N(N-1)$]	Complex multiplications $\left[\left(\frac{N}{2}\right)\log_2 N\right]$	Complex additions ($N\log_2 N$)
4	16	12	4	8
8	64	56	12	24
16	256	240	32	64
32	1024	992	80	160
64	4096	4032	192	384
128	16384	16256	448	896